Federated Conditional Stochastic Optimization

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Abstract

Conditional stochastic optimization has found applications in a wide range of machine learning tasks, such as invariant learning, AUPRC maximization, and meta-learning. As the demand for training models with large-scale distributed data grows in these applications, there is an increasing need for communication-efficient distributed optimization algorithms, such as federated learning algorithms. This paper considers the nonconvex conditional stochastic optimization algorithm (FCSG) with a conditional stochastic gradient estimator and a momentumbased algorithm (*i.e.* FCSG-M). To match the lower bound complexity in the single-machine setting, we design an accelerated algorithm (Acc-FCSG-M) via the variance reduction to achieve the best sample and communication complexity. Compared with the existing optimization analysis for MAML in FL, federated conditional stochastic optimization consider the sample of tasks. Extensive experimental results on various tasks validate the efficiency of these algorithms.

1 Introduction

The conditional stochastic optimization arises throughout a wide range of machine learning tasks, such as the policy evaluation in reinforcement learning [7], invariant learning [19], instrumental variable regression in causal inference [36], Model-Agnostic Meta-Learning (MAML) [13], AUPRC maximization [34] and so on. Recently many efficient conditional stochastic optimization algorithms have been developed [19–21, 34, 40, 38] to solve the corresponding machine learning problems and applications. However, all existing conditional stochastic optimization algorithms were only designed for centralized learning (*i.e.* model and data both deployed at a single machine) or finite-sum optimization, without considering the large-scale online distributed scenario. Many federated learning algorithms [31, 26, 35, 27, 46, 30, 22] were proposed as an communication-efficient training paradigm for large-scale machine learning training utilizing data from many worker nodes. In federated learning, worker nodes update the model locally, and the global server aggregates the model parameters periodically. Although federated learning has been actively applied to numerous real-world applications in the past years, the federated conditional stochastic optimization problem is still underexplored. To bridge this gap, in this paper we study the following federated conditional

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stochastic optimization problem:

$$\min_{x \in \mathcal{X}} F(x) := \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}_{\xi^n} f_{\xi^n}^n (\mathbb{E}_{\eta^n | \xi^n} g_{\eta^n}^n(x, \xi^n)),$$
(1)

where $\mathcal{X} \subseteq \mathbb{R}^d$ is a closed convex set, $\mathbb{E}_{\xi^n} f_{\xi^n}^n(\cdot) : \mathbb{R}^{d'} \to \mathbb{R}$ is the outer-layer function on the *n*-th device with the randomness ξ^n , and $\mathbb{E}_{\eta^n | \xi^n} g_{\eta^n}^n(\cdot, \xi^n) : \mathbb{R}^d \to \mathbb{R}^{d'}$ is the inner-layer function on the *n*-th device with respect to the conditional distribution of $\eta^n | \xi^n$. We assume $f_{\xi}^n(\cdot)$ and $g_{\eta}^n(\cdot, \xi)$ are continuously differentiable. The objective subsumes two stochastic functions in (1), where the inner functions rely on the randomnesses of both inner and outer layers, and ξ and η are not independent , which makes the federated conditional stochastic optimization more challenging compared with the standard federated learning optimization problems.

Federated conditional stochastic optimization contains the standard federated learning optimization as a special situation when the inner-layer function $g_{\eta^n}^n(x,\xi^n) = x$. In addition, federated stochastic compositional optimization is similar to federated conditional stochastic optimization given that both problems contain two-layer nested expectations. However, they are fundamentally different. In federated stochastic compositional optimization, the inner randomness η and the outer randomness ξ are independent and data samples of the inner layer are available directly from η (instead of a conditional distribution as in Problem (1)). Therefore, when randomnesses η and ξ are independent and $g_{\eta^n}^n(x, \cdot) = g_{\eta^n}^n(x)$, (1) is converted into federated stochastic compositional optimization [14].

Recently, to solve the conditional stochastic optimization problem efficiently, [19] studied the sample complexity of the sample average approximation for conditional stochastic optimization. Afterwards, [20] proposed the algorithm called biased stochastic gradient descent (BSGD) and an accelerated algorithm called biased SpiderBoost (BSpiderBoost). The convergence guarantees of BSGD and BSpiderBoost under different assumptions are established. More recently, [34, 40, 44, 38, 17] reformulated the AUC maximization into a finite-sum version of conditional stochastic optimization and introduced algorithms to solve it. In an increasing amount of distributed computing settings, efficient federated learning algorithms are absolutely necessary but still lacking. An important example of CSO is MAML. Instead of training a model [41, 4, 29, 10, 5, 32], in meta learning, we attempt to train models that can efficiently adapt to unseen tasks via learning with metadata from similar tasks [13]. When the tasks are distributed at different worker nodes, a federated version of MAML would be beneficial to leverage information from all workers [3]. A lot of existing efforts [23, 14]has been made to convert FL MAML into federated compositional optimization, which ignores the sample of tasks in MAML. Nonetheless, federated conditional stochastic optimization problems have never been studied. Thus, there exists a natural question: Can we design federated algorithms for conditional stochastic optimization with maintaining the fast convergence rate to solve Problem (1)?

In the paper, we give an affirmative answer to the above question. We propose a suite of approaches to solve Problem (1) and establish their corresponding convergence guarantee. To our best knowledge, this is the first work that thoroughly studies federated conditional stochastic optimization problems and provides completed theoretical analysis. Our proposed algorithm matches the lower-bound sample complexity in a single-machine setting and obtains convincing results in empirical experiments. Our main contributions are four-fold:

- 1) we propose federated conditional stochastic gradient (FCSG) algorithm to solve Problem (1). We establish the theoretical convergence analysis for FCSG. In the general nonconvex setting, we prove that FCSG has a sample complexity of $O(\epsilon^{-6})$ and communication complexity of $O(\epsilon^{-3})$ to reach an ϵ -stationary point, and achieves an appealing linear speedup *w.r.t* the number of worker nodes.
- 2) To further improve the empirical performances of our algorithm, we introduce a momentumbased FCSG algorithm, called FCSG-M since the momentum-based estimator could reduce noise from samples with history information. FCSG-M algorithm obtains the same theoretical guarantees as FCSG.
- 3) To reach the lower bound of sample complexity of the single-machine counterpart [20], we propose an accelerated version of FCSG-M (Acc-FCSG-M) based on the momentumbased variance reduction technique. We prove that Acc-FCSG-M has a sample complexity of $O(\epsilon^{-5})$, and communication complexity of $O(\epsilon^{-2})$, which matches the best sample complexity attained by single-machine algorithm BSpiderBoost with variance reduction.

4) Experimental results on the robust logistic regression, MAML and AUPRC maximization tasks validate the effectiveness of our proposed algorithms.

2 Related Work

2.1 Conditional Stochastic Optimization

[19] studied the generalization error bound and sample complexity of the sample average approximation (SAA) for conditional stochastic optimization. Subsequently, [20] proposed a class of efficient stochastic gradient-based methods for general conditional stochastic optimization to reach either a global optimal point in the convex setting or a stationary point in the nonconvex setting, respectively. In the nonconvex setting, BSGD has the sample complexity of $O(\epsilon^{-6})$ and a variance reduction algorithm (BSpiderBoost) has the sample complexity of $O(\epsilon^{-5})$. [21] utilized the Monte Carlo method to achieve better results compared with the vanilla stochastic gradient method. Recently, [34] converted AUPRC maximization optimization into the finite-sum version of the conditional stochastic optimization and propose adaptive and non-adaptive stochastic algorithms to solve it. Similarly, recent work [40] used moving average techniques to improve the convergence rate of AUPRC maximization optimization and provide theoretical analysis for the adaptive algorithm. Furthermore, [38] focused on finite-sum coupled compositional stochastic optimization, which limits the outer-layer function to the finite-sum structure. The algorithms proposed in [38] improved oracle complexity with the parallel speed-up. More recently, [17] use federated learning to solve AUC maximization. However, algorithms proposed in [34, 40, 38, 17] for AUC maximization have a significant limitation because they maintain an inner state for each data point. As a result, its convergence rate depends on the number of data points and cannot be extended to other tasks and large-scale model training. It is also not applicable to online learning due to the dependence on each local data point. [45] consider the decentralised online AUPRC maximization but the theoretical analysis cannot be applied into the federated learning.

2.2 Stochastic Compositional Optimization

Recently, a related optimization problem, stochastic compositional optimization, has attracted widely attention [43, 48, 14] and solve the following objective:

$$\min_{x \in \mathcal{X}} F(x) := \mathbb{E}_{\xi} f_{\xi}(\mathbb{E}_{\eta} g_{\eta}(x)) \,. \tag{2}$$

To address this problem, [43] developed SCGD, which utilizes the moving average technique to estimate the inner-layer function value. [42] further developed an accelerated SCGD method with the extrapolation-smoothing scheme. Subsequently, a series of algorithms [24, 50, 18, 48] were presented to improve the complexities using the acceleration or variance reduction techniques.

More recently, [23] and [14] studied the stochastic compositional problem in federated learning. [23] transformed the distributionally robust federated learning problem (*i.e.* a minimax optimization problem) into a simple compositional optimization problem by using KL divergence regularization and proposed the first federated learning compositional algorithm and analysis. [11] formulated the model personalization problem in federated learning as a model-agnostic meta-learning problem. In personalized federated learning, each client's task assignment is fixed and there is no task sampling on each client in the training procedure. The sampling of the inner layer and outer layer are independent. Therefore, personalized federated learning is formulated as the stochastic compositional optimization [23]. [39] solve personalized federated learning utilizing SCGD, in contrast to SGD in [23], to reduce the convergence complexities. However, the algorithm in [39] has a drawback in that keeping an inner state for each task is necessary, which is prohibitively expensive in large-scale settings. More recently, [14] proposed a momentum-like method for nonconvex problems with better complexities to solve the stochastic compositional problem in the federated learning setting. Although [14] claims their algorithm can be used in the MAML problem, it does not consider the two fundamental characteristics in MAML, i.e. task sampling and the dependency of inner data distribution on the sampled task.

Overall, problems (1) and (2) differ in two aspects: i) in Problem (2), the inner randomness η and the outer randomness ξ are independent, while in Problem (1), η is conditionally dependent on the ξ ; ii) in Problem (1), the inner function depends on both ξ and η . Therefore, Problem (2) can be regarded as a special case of (1). Thus the conditional stochastic optimization (1) is more general.

Table 1: Complexity summary of proposed federated conditional stochastic optimization algorithms to reach an ε -stationary point. Sample complexity is defined as the number of calls to the First-order Oracle (IFO) by worker nodes to reach an ε -stationary point. Communication complexity denotes the total number of back-and-forth communication rounds between each worker node and the central server required to reach an ε -stationary point.

Algorithm	Sample	Communication
FCSG	$O\left(\epsilon^{-6}\right)$	$O\left(\epsilon^{-3}\right)$
FCSG-M	$O\left(\epsilon^{-6}\right)$	$O\left(\epsilon^{-3}\right)$
Lower Bound [19]	$O\left(\epsilon^{-5}\right)$	-
Acc-FCSG-M	$O\left(\epsilon^{-5}\right)$	$O\left(\epsilon^{-2}\right)$

3 Preliminary

For solving the problem (1), we first consider the local objective $F^n(x)$ and its gradient. We have

$$F^n(x) = \mathbb{E}_{\xi^n} f^n_{\xi^n}(\mathbb{E}_{\eta^n | \xi^n} g^n_{\eta^n}(x, \xi^n))$$

$$\nabla F^n(x) = \mathbb{E}_{\xi^n} \left[\left(\mathbb{E}_{\eta^n | \xi^n} \nabla g^n_{\eta^n}(x, \xi^n) \right] \right)^\top \nabla f^n_{\xi^n}(\mathbb{E}_{\eta^n | \xi^n} g^n_{\eta^n}(x, \xi^n)) \right]$$

Since there are two layers of stochastic functions, the standard stochastic gradient estimator is not an unbiased estimation for the full gradient. Instead of constructing an unbiased stochastic gradient estimator, [20] considered a biased estimator of $\nabla F(x)$ using one sample ξ and m sample η :

$$\nabla \hat{F}^n\left(x;\xi^n,\mathcal{B}_n\right) = \left(\frac{1}{m}\sum_{\eta^n_j\in\mathcal{B}_n}\nabla g^n_{\eta^n_j}(x,\xi^n)\right)^\top \nabla f^n_{\xi^n}\left(\frac{1}{m}\sum_{\eta^n_j\in\mathcal{B}_n}g^n_{\eta^n_j}(x,\xi^n)\right)$$

where $\mathcal{B}_n = \left\{\eta_j^n\right\}_{j=1}^m$. And $\nabla \hat{F}^n(x;\xi^n,\mathcal{B}_n)$ is the gradient of an empirical objective such that

$$\hat{F}^n\left(x;\xi^n,\mathcal{B}_n\right) := f_{\xi^n}^n\left(\frac{1}{m}\sum_{\eta_j^n\in\mathcal{B}_n}g_{\eta_j^n}(x,\xi^n)\right) \tag{3}$$

3.1 Assumptions

Assumption 3.1. (Smoothness) $\forall n \in [N]$, the function $f_{\xi^n}^n(\cdot)$ is S_f -Lipschitz smooth, and the function $g_{\eta^n}^n(\cdot,\xi^n)$ is S_g -Lipschitz smooth, *i.e.*, for a sample ξ^n and m samples $\{\eta_j^n\}_{j=1}^m$ from the conditional distribution $P(\eta^n \mid \xi^n)$, $\forall x_1, x_2 \in \text{dom } f^n(\cdot)$, and $\forall y_1, y_2 \in \text{dom } g^n(\cdot)$, there exist $S_f > 0$ and $S_g > 0$ such that

$$\mathbb{E}\|\nabla f_{\xi^n}^n(x_1) - \nabla f_{\xi^n}^n(x_2)\| \le S_f \|x_1 - x_2\| \quad \mathbb{E}\|\nabla g_{\eta^n}^n(y_1, \xi^n) - \nabla g_{\eta^n}^n(y_2, \xi^n)\| \le S_g \|y_1 - y_2\|$$

Assumption 3.1 is a widely used assumption in optimization analysis. Many single-machine stochastic algorithms use this assumption, such as BSGD [20], SPIDER [12], STORM [6], ADSGD [1], and D2SG [15]. In distributed learning, the convergence analysis of distributed learning algorithms, such as DSAL [2], and many federated learning algorithms such as MIME [25], Fed-GLOMO [8], and STEM [27] also depend on it.

Assumption 3.2. (Bounded gradient) $\forall n \in [N]$, the function $f^n(\cdot)$ is L_f -Lipschitz continuous, and the function $g^n(\cdot)$ is L_g -Lipschitz continuous, *i.e.*, $\forall x \in \text{dom } f^n(\cdot)$, and $\forall y \in \text{dom } g^n(\cdot)$, the second moments of functions are bounded as below:

$$\mathbb{E} \|\nabla f_{\xi^n}^n(x)\|^2 \le L_f^2 \quad \mathbb{E} \|\nabla g_{\eta^n}^n(y_1,\xi^n)\|^2 \le L_g^2$$

Assumption 3.2 is a typical assumption in the multi-layer problem optimization to constrain the upper bound of the gradient of each layer, as in [43, 34, 14, 17].

Assumption 3.3. (Bounded variance) [20] $\forall n \in [N]$, and $x \in \mathcal{X}$:

$$\sup_{\xi^n, x \in \mathcal{X}} \mathbb{E}_{\eta^n | \xi^n} \| g_{\eta^n}(x, \xi^n) - \mathbb{E}_{\eta^n | \xi^n} g_{\eta^n}(x, \xi^n) \|^2 \le \sigma_g^2$$

where $\sigma_q^2 < +\infty$. Assumption 3.3 indicates that the random vector g_{η^n} has bounded variance.

Algorithm 1 FCSG and FCSG-M Algorithm

1: Input: Parameters: T, momentum weight β , learning rate α , the number of local updates q, inner

batch size *m* and outer batch size *b*, as well as the initial outer batch size B; 2: **Initialize:** $x_0^n = \bar{x}_0 = \frac{1}{N} \sum_{k=1}^N x_0^n$. Draw *B* samples of $\{\xi_{t,1}^n, \dots, \xi_{t,B}^n\}$ and draw *m* samples $\mathcal{B}_{0,i}^n = \{\eta_{ij}^n\}_{j=1}^m$ from $P(\eta^n \mid \xi_{0,i}^n)$ for each $\xi_{0,i}^n \in \{\xi_{t,1}^n, \dots, \xi_{t,B}^n\}; u_1^n = \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{j=1}$ $\frac{\frac{1}{B}\sum_{i=1}^{B}\nabla \hat{F}^{n}(x_{0}^{n};\xi_{0,i}^{n},\mathcal{B}_{0,i}^{n}).$ 3: **for** $t=1,2,\ldots,T$ **do** for n = 1, 2, ..., N do 4: 5: if mod(t,q) = 0 then Server Update: 6: $u_{t}^{n} = \bar{u}_{t} = \frac{1}{N} \sum_{i=1}^{N} u_{t}^{n}$ $x_{t}^{n} = \bar{x}_{t} = \frac{1}{N} \sum_{n=1}^{N} (x_{t-1}^{n} - \alpha u_{t}^{n})$ else 7: 8: $x_t^n = x_{t-1}^n - \alpha u_t^n$ end if 9: 10: 11: Draw *b* samples of $\{\xi_{t,1}^n, \dots, \xi_{t,b}^n\}$ Draw *m* samples $\mathcal{B}_{t,n}^n = \{\eta_{ij}^n\}_{j=1}^m$ from $P(\eta^n \mid \xi_{t,i}^n)$ for each $\xi_{t,i}^n \in \{\xi_{t,1}^n, \dots, \xi_{t,b}^n\}$, 12: 13: $u_{t+1}^n = \frac{1}{b} \sum_{i=1}^b \nabla \hat{F}^n(x_t^n; \xi_{t,i}^n, \mathcal{B}_{t,i}^n)$ 14: $u_{t+1}^{n} = (1-\beta)u_{t}^{n} + \frac{\beta}{b}\sum_{i=1}^{b}\nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n})$ 15: end for 16: 17: end for 18: **Output:** x chosen uniformly random from $\{\bar{x}_t\}_{t=1}^T$.

4 **Proposed Algorithms**

In the section, we propose a class of federated first-order methods to solve the Problem (1). We first design a federated conditional stochastic gradient (FCSG) algorithm with a biased gradient estimator and the momentum-based algorithm FCSG-M. To further accelerate our algorithm and achieve the lower bound of sample complexity of the single-machine algorithm, we design the Acc-FCSG-M with a variance reduction technique. Table 2 summarizes the complex details of each algorithm.

4.1 Federated Conditional Stochastic Gradient (FCSG)

First, we design a federated conditional stochastic gradient (FCSG) algorithm with the biased gradient estimator. We leverage a mini-batch of conditional samples to construct the gradient estimator u_t with controllable bias as (6). At each iteration, worker nodes update their local models x_t with local data, which can be found in Lines 8-13 of Algorithm 1. Once every q local iterations, the server collects local models and returns the averaged models to each worker node, as Lines 5-7 of Algorithm 1. Here, the number of local update steps q is greater than 1 such that the number of communication rounds is reduced to T/q. The details of the method are summarized in Algorithm 1. Then we study the convergence properties of our new algorithm FCSG. Detailed proofs are provided in the supplementary materials.

Theorem 4.1. Suppose Assumptions 3.1, 3.2 and 3.3 hold, if $\alpha \leq \frac{1}{6qS_F}$, FCSG has the following convergence result

$$\frac{1}{T}\sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \le \frac{2[F(\bar{x}_0) - F(\bar{x}_T)]}{\alpha T} + \frac{2L_g^2 S_f^2 \sigma_g^2}{m} + \frac{2\alpha S_F L_f^2 L_g^2}{N} + 42(q-1)q\alpha^2 L_f^2 L_g^2 S_F^2$$

Corollary 4.2. We choose $\alpha = \frac{1}{6S_F} \sqrt{\frac{N}{T}}$ and $q = (T/N^3)^{1/4}$, we have

$$\frac{1}{T}\sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \le \frac{12S_F[F(\bar{x}_0) - F(\bar{x}_*)]}{(NT)^{1/2}} + \frac{2L_g^2 S_f^2 \sigma_g^2}{m} + \frac{L_f^2 L_g^2}{6(NT)^{1/2}} + \frac{19L_f^2 L_g^2}{9(NT)^{1/2}}$$

Remark 4.3. We choose $B = b = O(1) \ge 1$, and $m = O(\varepsilon^{-2})$, according to Corollary 4.2 to let $\frac{1}{T} \sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \le \varepsilon^2$, we get $T = O(N^{-1}\varepsilon^{-4})$. $O(N^{-1}\varepsilon^{-4})$ indicates the linear speedup of our algorithm. Given $q = (T/N^3)^{1/4}$, the communication complexity is $\frac{T}{q} = (NT)^{3/4} = O(\varepsilon^{-3})$. Then the sample complexity is $mT = O(N^{-1}\varepsilon^{-6})$.

4.2 Federated Conditional Stochastic Gradient with Momentum (FCSG-M)

Next, we propose a momentum-based local updates algorithm (FCSG-M) for federated conditional stochastic optimization problems. Momentum is a popular technique widely used in practice for training deep neural networks. The motivation behind it in local updates is to use the historic information (*i.e.*, averaging of stochastic gradients) to reduce the effect of stochastic gradient noise. The details of our method are shown in the Algorithm 1.

Initially, each device utilizes the standard stochastic gradient descent method to update the model parameter, as seen in Line 2 of Algorithm 1. Afterward, compared with FCSG, at each step, each client uses momentum-based gradient estimators u_t to update the local model, which can be found in Lines 9-14 of Algorithm 1. The coefficient β for the update of u_t should satisfy $0 < \beta < 1$. Every q iteration, the clients communicate $\{x_t, u_t\}$ to the server, which computes the $\{\bar{x}_t, \bar{u}_t\}$, and returns the averaged model and gradient estimator to each worker node, as Lines 5-7 of Algorithm 1. Then we study the convergence properties of our new algorithm FCSG-M. The details of the proof are provided in the supplementary materials.

Theorem 4.4. Suppose Assumptions 3.1, 3.2 and 3.3 hold, $\alpha \leq \frac{1}{6qS_F}$ and $\beta = 5S_F\eta$. FCSG-M has the following convergence result

$$\begin{split} &\frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\|\nabla F(\bar{x}_t)\|^2 \leq 2\frac{F(\bar{x}_0) - F(\bar{x}_T)}{\alpha T} \\ &+ \frac{96S_F^2}{5}q^2\alpha^2 [L_f^2 L_g^2(1+\frac{1}{N}) + 3L_f^2 L_g^2] + \frac{4L_g^2 S_f^2 \sigma_g^2}{m} + \frac{8L_f^2 L_g^2}{\beta BT} + \frac{8\beta L_f^2 L_g^2}{N} \end{split}$$

Corollary 4.5. We choose $\alpha = \frac{1}{6S_F} \sqrt{\frac{N}{T}}$, $q = (T/N^3)^{1/4}$, we have

$$\begin{split} \frac{1}{T} \sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 &\leq \frac{12S_F[F(\bar{x}_0) - F(\bar{x}_*)]}{(NT)^{1/2}} \\ &+ \frac{112L_f^2 L_g^2}{3(NT)^{1/2}} + \frac{4L_g^2 S_f^2 \sigma_g^2}{m} + \frac{48L_f^2 L_g^2}{5(NT)^{1/2}} + \frac{20L_f^2 L_g^2}{3(NT)^{1/2}} \end{split}$$

Remark 4.6. We choose b = O(1), B = O(1), and $m = O(\varepsilon^{-2})$. According to Corollary 4.5 to make $\frac{1}{T} \sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \le \varepsilon^2$, we get $T = O(N^{-1}\varepsilon^{-4})$. Given $q = (T/N^3)^{1/4}$, the communication complexity is $\frac{T}{q} = (NT)^{3/4} = O(\varepsilon^{-3})$. The sample complexity is $mT = O(N^{-1}\varepsilon^{-6})$, which indicates FCSG-M also has the linear speedup with respect to the number of worker nodes.

4.3 Acc-FCSG-M

In the single-machine setting, [20] presents that under the general nonconvex conditional stochastic optimization objective, the lower bound of sample complexity is $O(\varepsilon^{-5})$. It means, the sample complexity achieved by FCSG and FCSG-M could be improved for nonconvex smooth conditional stochastic optimization objectives. To match the above lower bound of sample complexity, we propose an accelerated version of the FCSG-M (Acc-FCSG-M) based on the momentum-based variance reduction technique. The details of the method are shown in Algorithm 2.

Similar to the FCSG-M, in the beginning, each worker node initializes the model parameters and utilizes the stochastic gradient descent method to calculate the gradient estimator. Subsequently, in every q iteration, all worker nodes perform communication with the central server, and the model parameters and gradient estimators are averaged. The key difference is that at step 13 in Acc-FCSG-M, we use the momentum-based variance reduction gradient estimator u_{t+1}^n to track the gradient and update the model. where $\beta \in (0, 1)$. We establish the theoretical convergence guarantee of our new algorithm Acc-FCSG-M. All proofs are provided in the supplementary materials.

Algorithm 2 Acc-FCSG-M Algorithm

1: **Input:** T, momentum weight β , learning rate α , the number of local updates q, inner batch size

m and outer batch size *b*, as well as the initial outer batch size B; 2: **Initialize:** $x_0^n = \frac{1}{N} \sum_{k=1}^N x_0^n$. Draw *B* samples of $\{\xi_1^n, \dots, \xi_B^n\}$ and draw *m* samples $\mathcal{B}_{0,i}^n = \{\eta_{ij}^n\}_{j=1}^m$ from $P(\eta^n \mid \xi_i^n)$ for each $\xi_i^n \in \{\xi_1^n, \dots, \xi_B^n\}$, then $u_1^n = \sum_{k=1}^n e^{-1} f(\xi_1^n, \dots, \xi_B^n)$. $\frac{\frac{1}{B}\sum_{i=1}^{B}\nabla \hat{F}^{n}(x_{0}^{n};\xi_{0,i}^{n},\mathcal{B}_{0,i}^{n}) \text{ for } n \in [N].$ 3: for $t = 1, 2, \ldots, T$ do for n = 1, 2, ..., N do 4: $\begin{array}{l} \underset{t}{\underset{t}{\text{mod}}} & (t,q) = 0 \text{ then} \\ \textbf{Server Update:} \\ & u_t^n = \bar{u}_t = \frac{1}{N} \sum_{i=1}^N u_t^n \\ & x_t^n = \bar{x}_t = \frac{1}{N} \sum_{n=1}^N (x_{t-1}^n - \alpha u_t^n) \\ \textbf{else} \end{array}$ 5: 6: 7: 8: 9: $x_{t,n} = x_{t-1}^n - \alpha u_t^n$ end if 10: 11: Draw *b* samples of $\{\xi_{t,1}^n, \dots, \xi_{t,b}^n\}$ Draw *m* samples $\mathcal{B}_{t,n}^n = \{\eta_{ij}^n\}_{j=1}^m$ from $P(\eta^n \mid \xi_{t,i}^n)$ for each $\xi_{t,i}^n \in \{\xi_{t,1}^n, \dots, \xi_{t,B}^n\}$, $u_{t+1}^n = \frac{1}{b} \sum_{i=1}^b \nabla \hat{F}^n(x_t^n; \xi_{t,i}^n, \mathcal{B}_{t,i}^n) + (1 - \beta)(u_t^n - \frac{1}{b} \sum_{i=1}^b \nabla \hat{F}^n(x_{t-1}^n; \xi_{t,i}^n, \mathcal{B}_{t,i}^n))$ end for 12: 13: 14: 15: 16: end for 17: **Output:** x chosen uniformly random from $\{\bar{x}_t\}_{t=1}^T$.

Theorem 4.7. Suppose Assumptions 3.1, 3.2 and 3.3 hold, $\alpha \leq \frac{1}{6gS_F}$ and $\beta = 5S_F \alpha$. Acc-FCSG-M has the following

$$\frac{1}{T}\sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \leq \frac{2[F(\bar{x}_0) - F(\bar{x}_T)]}{T\alpha} + \frac{3L_f^2 L_g^2}{\beta BNT} + \frac{13L_f^2 L_g^2 c^2}{6S_F^2}\alpha^2 + \frac{3L_g^2 S_f^2 \sigma_g^2}{m} + \frac{6\beta L_f^2}{Nb} + \frac{13L_f^2 L_g^2 c^2}{\delta S_F^2}\alpha^2 + \frac{3L_g^2 S_f^2 \sigma_g^2}{\delta S_F^2} + \frac{3L_g^2 S_F^2 \sigma_g^2} + \frac{3L_g^2 S_F^2 \sigma_g^2} + \frac{3L_g^2 S_F^2 \sigma_g^2}{\delta S_F^2} + \frac{$$

Corollary 4.8. We choose $q = (T/N^2)^{1/3}$. Therefore, $\alpha = \frac{1}{12qS_F} = \frac{N^{2/3}}{12S_FT^{1/3}}$, since $c = \frac{30S_F^2}{bN}$, we have $\beta = c\alpha^2 = \frac{5N^{1/3}}{24T^{2/3}b}$. And let $B = \frac{T^{1/3}}{N^{2/3}}$. Therefore, we have

$$\begin{split} \frac{1}{T} \sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 &\leq \frac{24S_F[F(\bar{x}_0) - F(\bar{x}_*)]}{(NT)^{2/3}} \\ &+ \frac{72L_f^2 L_g^2 b}{5(NT)^{2/3}} + \frac{325L_f^2 L_g^2}{24b^2(TN)^{2/3}} + \frac{3L_g^2 S_f^2 \sigma_g^2}{m} + \frac{5L_f^2}{4b^2(NT)^{2/3}} \end{split}$$

Remark 4.9. We choose b as $O(1)(b \ge 1)$ and $m = O(\varepsilon^{-2})$. According to Corollary 4.8 to make $\frac{1}{T}\sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \leq \varepsilon^2, \text{ we get } T = O(N^{-1}\varepsilon^{-3}) \text{ and } \frac{T}{q} = (NT)^{2/3} = O(\varepsilon^{-2}). \text{ The sample complexity is } O(N^{-1}\varepsilon^{-5}). \text{ The communication complexity is } O(\varepsilon^{-2}). T = O(N^{-1}\varepsilon^{-3}) \text{ indicates } O(N^{-1}\varepsilon^{-3}).$ the linear speedup of our algorithm.

5 **Experiments**

In this section, we conduct experiments to validate the efficiency of our algorithms on two machine learning tasks: 1) Invariant Logistic Regression, and 2) Federated Model-Agnostic Meta-Learning. The experiments are run on CPU machines with AMD EPYC 7513 32-Core Processors as well as NVIDIA RTX A6000.

5.1 Invariant Logistic Regression

Invariant learning has an important role in robust classifier training [33]. In this section, We compare the performance of our algorithms, FCSG and FCSG-M, on the distributed invariant logistic regression

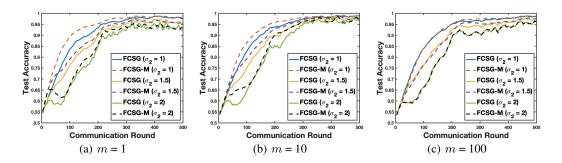


Figure 1: Test accuracy vs the number of communication rounds for different inner mini-batch (m = 1, 10, 100) under different noise ratios ($\sigma_2/\sigma_1 = 1, 1.5, 2$).

to evaluate the benefit from momentum and the effect of inner batch size, and the problem was formulated by [19]:

$$\min_{x} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}_{\xi^{n} = (a,b)} [l_{n}(x) + g(x)]$$
where $l_{n}(x) = \log(1 + \exp(-b\mathbb{E}_{\eta^{n}|\xi^{n}}[(\eta^{n})^{\top}x])) \quad g(x) = \lambda \sum_{i=1}^{d} \frac{\gamma x_{i}^{2}}{1 + \gamma x_{i}^{2}}$
(4)

where $l_n(x)$ is the logistic loss function and g(x) is a non-convex regularization. We follow the experimental protocols in [20] and set the dimension of the model as 10 over 16 worker nodes. We construct the dataset $\xi^n = (a, b)$ and η as follow: We sample $a \sim N(0, \sigma_1^2 I_d)$, set $b = \{\pm 1\}$ according to the sign of $a^{\top}x^*$, then sample $\eta_{ij}^n \sim N(a, \sigma_2^2 I_d)$. We choose $\sigma_1 = 1$, and consider the σ_2/σ_1 from $\{1, 1.5, 2\}$. At each local iteration, we use a fixed mini-batch size m from $\{1, 10, 100\}$. The outer batch size is set as 1. We test the model with 50000 outer samples to report the test accuracy. We carefully tune hyperparameters for both methods. $\lambda = 0.001$ and $\alpha = 10$. We run a grid search for the learning rate and choose the learning rate in the set $\{0.01, 0.005, 0.001\}$. β in FCSG-M are chosen from the set $\{0.001, 0.01, 0.1, 0.5, 0.9\}$. The local update step is set as 50. The experiments are presented in Figure 1.

Figure 1 shows that when the noise ratio σ_2/σ_1 increases, larger inner samples *m* are needed, as suggested by the theory because a large batch size could reduce sample noise. In addition, when m = 100, FCSG and FCSG-M have similar performance. However, when batch size is small, compared with FCSG, FCSG-M has a more stable performance, because FCSG-M can use historic information to reduce the effect of stochastic gradient noise.

5.2 Federated Model-Agnostic Meta-Learning

Next, we evaluate our proposed algorithms with the few shot image classification task over the dataset with baselines: Local-SCGD, Local-SCGDM [14]. MOML [39] is not suitable for this task since MOML requires maintaining an inner state for each task which is not permitted due to the large number of tasks in the Omniglot dataset. Local-SCGD is the federated version of SCGD [43]. This task can be effectively solved via Model-Agnostic Meta-Learning [13].

Meta-learning aims to train a model on various learning tasks, such that the model can easily adapt to a new task using few training samples. Model-agnostic meta-learning (MAML) [13] is a popular meta-learning method to learn a good initialization with a gradient-based update, which can be formulated as the following conditional stochastic optimization problem:

$$\min_{x} \mathbb{E}_{i \sim \mathcal{P}_{\text{task}}, a \sim D_{\text{query}}^{i}} \mathcal{L}_{i} \left(\mathbb{E}_{b \sim D_{\text{support}}^{i}} \left(x - \lambda \nabla \mathcal{L}_{i}(x, b) \right), a \right)$$

where $\mathcal{P}_{\text{task}}$ denotes the learning tasks distribution, D_{support}^i and D_{query}^i are support (training) dataset and query (testing) dataset of the learning task *i*, respectively. $\mathcal{L}_i(\cdot, D_i)$ is the loss function on dataset D_i of task *i*. And the λ is a fixed meta step size. Assume $\xi = (i, a)$ and $\eta = b$, the MAML problem is an example of conditional stochastic optimization where the support (training) samples in the inner

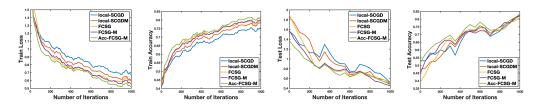


Figure 2: Convergence results of the 5-way-1-shot case over Omniglot Dataset.

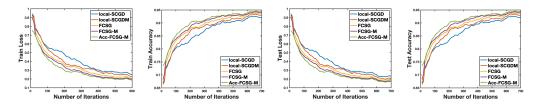


Figure 3: Convergence results of the 5-way-5-shot case over Omniglot Dataset.

layer for the meta-gradient update are drawn from the conditional distribution of $P(\eta \mid \xi)$ based on the sampled task in the outer layer.

Given there are a large number of pre-train tasks in MAML, federated learning is a good training strategy to improve efficiency because we can evenly distribute tasks over various worker nodes and the global server coordinates worker nodes to learn a good initial model collaboratively like MAML. Therefore, in Federated MAML, it is assumed that each device has part of the tasks. The optimization problem is defined as follows:

$$\min_{x \in \mathbb{R}^d} \frac{1}{N} \sum_{n=1}^N F^n(x) \triangleq \frac{1}{N} \sum_{n=1}^N f^n\left(g^n(x)\right)$$
where $g^n(x) = \mathbb{E}_{\eta^n \sim \mathcal{D}_{i, \text{ support}}^n}\left[x - \lambda \nabla \mathcal{L}_i^n\left(x; \eta^n\right)\right], f^n(y) = \mathbb{E}_{i \sim \mathcal{P}_{\text{task}}^n, a^n \sim \mathcal{D}_{i, \text{ guery}}^n} \mathcal{L}_i^n\left(y; a^n\right).$
(5)

In this part, we apply our methods to few-shot image classification on the Omniglot [28, 13]. The Omniglot dataset contains 1623 different handwritten characters from 50 different alphabets and each of the 1623 characters consists of 20 instances drawn by different persons. We divide the characters to train/validation/test with 1028/172/423 by Torchmeta [9] and tasks are evenly partitioned into disjoint sets and we distribute tasks randomly among 16 worker nodes. We conduct the fast learning of N-way-K-shot classification following the experimental protocol in [37]. The N-way-K-shot classification denotes we sample N unseen classes, randomly provide the model with K different instances of each class for training, and evaluate the model's ability to classify with new instances from the same N classes. We sample 15 data points for validation. We use a 4-layer convolutional neural model where each layer has 3 × 3 convolutions and 64 filters, followed by a ReLU nonlinearity and batch normalization [13]. The images from Omniglot are downsampled to 28×28 . For all methods, the model is trained using a single gradient step with a learning rate of 0.4. The model was evaluated using 3 gradient steps [13]. Then we use grid search and carefully tune other hyper-parameters for each method. We choose the learning rate from the set $\{0.1, 0.05, 0.01\}$ and η in as 1 [14]. We select the inner state momentum coefficient for Local-SCGD and Local-SCGDM from $\{0.1, 0.5, 0.9\}$ and outside momentum coefficient for Local-SCGDM, FCSG-M and Acc-FCSG-M from $\{0.1, 0.5, 0.9\}$.

Figures 2 and 3 show experimental results in the 5-way-1-shot and 5-way-5-shot cases, respectively. Results show that our algorithms outperform baselines by a large margin. The main reason for Local-SCGD and Local-SCGDM to have bad performance is that converting the MAML optimization to the stochastic compositional optimization is unreasonable. It ignores the effect of task sampling on the training and inner training data distribution changes based on the sampled tasks in the outer layer. The use of the momentum-like inner state to deal with the MAML will slow down the convergence and we have to tune the extra momentum coefficient for the inner state. In addition, the momentum-like inner state also introduces extra communication costs because the server needs to average the inner state as in Local-SCGDM. In addition, comparing the results in Figures 2 and 3, we can see when

Table 2: Final averaged AP scores on the testing data.

Datasets	FedAvg	CODA+	FCSG	FCSG-M	Acc-FCSG-M
MNIST	0.9357	0.9733	0.9868	0.9878	0.9879
CIFAR-10	0.5059	0.6039	0.7130	0.7157	0.7184

the K increases in the few-shot learning, the training performance is improved, which matches the theoretical analysis that a large inner batch-size *m* benefits the model training.

5.3 Federated Online AUPRC maximization

AUROC maximization in FL have been studied in [16, 49, 47] and AUPRC maximization is also used to solve the imbalanced classification. Existing AUPRC algorithms maintain an inner state for each data point. [45] consider the online AUPRC in the decentralized learning. For the large-scale distributed data over multiple clients, algorithms for online AUPRC maximization in FL is necessary.

Following [34] and [45], the surrogate function of average precision (AP) for online AUPRC maximization is:

$$\hat{AP} = \mathbb{E}_{\xi^+ \sim \mathcal{D}^+} \frac{\mathbb{E}_{\xi \sim \mathcal{D}} \mathbf{I} \left(y = 1 \right) \ell \left(x; z^+, z \right)}{\mathbb{E}_{\xi \sim \mathcal{D}} \quad \ell \left(x; z^+, z \right)}$$

where $\ell(x; z^+, z) = (\max\{s - h(x; z^+) + h(x; z), 0\})^2$ and h(x; z) is the prediction score function of input z with model x. Federated Online AUPRC maximization could be reformulated as:

$$\min_{x} F(x) = \min_{x} \frac{1}{N} \sum_{n=1}^{n} \mathbb{E}_{\xi_n \sim \mathcal{D}_n^+} f(\mathbb{E}_{\xi'_n \sim \mathcal{D}_n} g^n(\mathbf{x}; \xi_n, \xi'_n))$$
(6)

where $\xi_n = (z_n, y_n) \sim \mathcal{D}_n$ and $\xi_n^+ = (z_n^+, y_n^+) \sim \mathcal{D}_n^+$ are samples drawn from the whole datasets and positive datasets, respectively. It is a two-level problem and the inner objective depends on both ξ and ξ^+ . Since federated online AUPRC is a special example of a federated CSO, our algorithms could be directly applied to it.

We choose MNIST dataset and CIFAR-10 datasets. As AUROC maximization in federated settings has been demonstrated in existing works, [16, 49], we use CODA+ in [49] as a baseline. Another baseline is the FedAvg with cross-entropy loss. Since AUPRC is used for binary classification, the first half of the classes in the MNIST and CIFAR10 datasets are designated to be a negative class, and the rest half of the classes are considered to be the positive class. Then, we remove 80% of the positive examples in the training set to make it imbalanced, while keeping the test set unchanged. The results show that our algorithms could be used to solve the online AUPRC maximization in FL and it largely improves the model's performance.

6 Conclusion

In this paper, we studied federated conditional stochastic optimization under the general nonconvex setting. To the best of our knowledge, this is the first paper proposing algorithms for the federated conditional stochastic optimization problem. We first used the biased stochastic first-order gradient to design an algorithm called FCSG, which we proved to have a sample complexity of $O(\epsilon^{-6})$, and communication complexity of $O(\epsilon^{-3})$ to reach an ϵ -stationary point. FCSG enjoys an appealing linear speedup with respect to the number of worker nodes. To improve the empirical performances of FCSG, we also proposed a novel algorithm (*i.e.*, FCSG-M), which achieves the same theoretical guarantees as FCSG. To fill the gap from lower-bound complexity, we introduced an accelerated version of FCSG-M, called Acc-FCSG-M, using variance reduction technique, which is optimal for the nonconvex smooth federated conditional stochastic optimization problems as it matches the best possible complexity result achieved by the single-machine method. It has a sample complexity of $O(\epsilon^{-5})$, and communication complexity of the nonconvex optimization in federated learning. Experimental results on the machine learning tasks validate the effectiveness of our algorithms.

Limitation Conditional stochastic optimization has much broader applications and a more comprehensive evaluation of our proposed algorithms on other use cases would be a promising future direction. In addition, FCSG-M and accelerated FCSG-M require higher communication overhead.

References

- [1] Runxue Bao, Bin Gu, and Heng Huang. An accelerated doubly stochastic gradient method with faster explicit model identification. In *Proceedings of the 31st ACM International Conference on Information & Knowledge Management*, pages 57–66, 2022.
- [2] Runxue Bao, Xidong Wu, Wenhan Xian, and Heng Huang. Doubly sparse asynchronous learning for stochastic composite optimization. In *Thirty-First International Joint Conference on Artificial Intelligence* (*IJCAI*), pages 1916–1922, 2022.
- [3] Fei Chen, Mi Luo, Zhenhua Dong, Zhenguo Li, and Xiuqiang He. Federated meta-learning with fast convergence and efficient communication. arXiv preprint arXiv:1802.07876, 2018.
- [4] Jiangce Chen. *Representation-agnostic Shape Matching of 3D Models with Graph Machine Learning Methods*. PhD thesis, University of Connecticut, 2021.
- [5] Jiangce Chen, Horea T Ilies, and Caiwen Ding. Graph-based shape analysis for heterogeneous geometric datasets: Similarity, retrieval and substructure matching. *Computer-Aided Design*, 143:103125, 2022.
- [6] Ashok Cutkosky and Francesco Orabona. Momentum-based variance reduction in non-convex sgd. *Advances in neural information processing systems*, 32, 2019.
- [7] Bo Dai, Niao He, Yunpeng Pan, Byron Boots, and Le Song. Learning from conditional distributions via dual embeddings. In *Artificial Intelligence and Statistics*, pages 1458–1467. PMLR, 2017.
- [8] Rudrajit Das, Anish Acharya, Abolfazl Hashemi, Sujay Sanghavi, Inderjit S Dhillon, and Ufuk Topcu. Faster non-convex federated learning via global and local momentum. In *Uncertainty in Artificial Intelligence*, pages 496–506. PMLR, 2022.
- [9] Tristan Deleu, Tobias Würfl, Mandana Samiei, Joseph Paul Cohen, and Yoshua Bengio. Torchmeta: A Meta-Learning library for PyTorch, 2019. URL https://arxiv.org/abs/1909.06576. Available at: https://github.com/tristandeleu/pytorch-meta.
- [10] Jason Xiaotian Dou, Minxue Jia, Nika Zaslavsky, Runxue Bao, Shiyi Zhang, Ke Ni, Paul Pu Liang, Haiyi Mao, and Zhihong Mao. Learning more effective cell representations efficiently. In *NeurIPS 2022 Workshop on Learning Meaningful Representations of Life*, 2022. URL https://openreview.net/forum?id=oRULd-eaNZH.
- [11] Alireza Fallah, Aryan Mokhtari, and Asuman Ozdaglar. Personalized federated learning: A meta-learning approach. *arXiv preprint arXiv:2002.07948*, 2020.
- [12] Cong Fang, Chris Junchi Li, Zhouchen Lin, and Tong Zhang. Spider: Near-optimal non-convex optimization via stochastic path-integrated differential estimator. *Advances in Neural Information Processing Systems*, 31, 2018.
- [13] Chelsea Finn, Pieter Abbeel, and Sergey Levine. Model-agnostic meta-learning for fast adaptation of deep networks. In *International conference on machine learning*, pages 1126–1135. PMLR, 2017.
- [14] Hongchang Gao, Junyi Li, and Heng Huang. On the convergence of local stochastic compositional gradient descent with momentum. In *International Conference on Machine Learning*, pages 7017–7035. PMLR, 2022.
- [15] Bin Gu, Runxue Bao, Chenkang Zhang, and Heng Huang. New scalable and efficient online pairwise learning algorithm. *IEEE Transactions on Neural Networks and Learning Systems*, 2023.
- [16] Zhishuai Guo, Mingrui Liu, Zhuoning Yuan, Li Shen, Wei Liu, and Tianbao Yang. Communication-efficient distributed stochastic auc maximization with deep neural networks. In *International conference on machine learning*, pages 3864–3874. PMLR, 2020.
- [17] Zhishuai Guo, Rong Jin, Jiebo Luo, and Tianbao Yang. Fedxl: Provable federated learning for deep x-risk optimization. 2022.
- [18] Wenqing Hu, Chris Junchi Li, Xiangru Lian, Ji Liu, and Huizhuo Yuan. Efficient smooth non-convex stochastic compositional optimization via stochastic recursive gradient descent. Advances in Neural Information Processing Systems, 32, 2019.
- [19] Yifan Hu, Xin Chen, and Niao He. Sample complexity of sample average approximation for conditional stochastic optimization. SIAM Journal on Optimization, 30(3):2103–2133, 2020.

- [20] Yifan Hu, Siqi Zhang, Xin Chen, and Niao He. Biased stochastic first-order methods for conditional stochastic optimization and applications in meta learning. *Advances in Neural Information Processing Systems*, 33:2759–2770, 2020.
- [21] Yifan Hu, Xin Chen, and Niao He. On the bias-variance-cost tradeoff of stochastic optimization. Advances in Neural Information Processing Systems, 34:22119–22131, 2021.
- [22] Zhengmian Hu, Xidong Wu, and Heng Huang. Beyond lipschitz smoothness: A tighter analysis for nonconvex optimization. 2023.
- [23] Feihu Huang, Junyi Li, and Heng Huang. Compositional federated learning: Applications in distributionally robust averaging and meta learning. arXiv preprint arXiv:2106.11264, 2021.
- [24] Zhouyuan Huo, Bin Gu, Ji Liu, and Heng Huang. Accelerated method for stochastic composition optimization with nonsmooth regularization. In *Thirty-Second AAAI Conference on Artificial Intelligence*, 2018.
- [25] Sai Praneeth Karimireddy, Martin Jaggi, Satyen Kale, Mehryar Mohri, Sashank J Reddi, Sebastian U Stich, and Ananda Theertha Suresh. Mime: Mimicking centralized stochastic algorithms in federated learning. arXiv preprint arXiv:2008.03606, 2020.
- [26] Sai Praneeth Karimireddy, Satyen Kale, Mehryar Mohri, Sashank Reddi, Sebastian Stich, and Ananda Theertha Suresh. Scaffold: Stochastic controlled averaging for federated learning. In *International conference on machine learning*, pages 5132–5143. PMLR, 2020.
- [27] Prashant Khanduri, Pranay Sharma, Haibo Yang, Mingyi Hong, Jia Liu, Ketan Rajawat, and Pramod Varshney. Stem: A stochastic two-sided momentum algorithm achieving near-optimal sample and communication complexities for federated learning. *Advances in Neural Information Processing Systems*, 34: 6050–6061, 2021.
- [28] Brenden Lake, Ruslan Salakhutdinov, Jason Gross, and Joshua Tenenbaum. One shot learning of simple visual concepts. In *Proceedings of the annual meeting of the cognitive science society*, volume 33, 2011.
- [29] Jizhizi Li, Jing Zhang, Stephen J Maybank, and Dacheng Tao. Bridging composite and real: towards end-to-end deep image matting. *International Journal of Computer Vision*, 130(2):246–266, 2022.
- [30] Tian Li, Anit Kumar Sahu, Manzil Zaheer, Maziar Sanjabi, Ameet Talwalkar, and Virginia Smith. Federated optimization in heterogeneous networks. *Proceedings of Machine learning and systems*, 2:429–450, 2020.
- [31] Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Aguera y Arcas. Communication-efficient learning of deep networks from decentralized data. In *Artificial intelligence and statistics*, pages 1273–1282. PMLR, 2017.
- [32] Yongsheng Mei, Hanhan Zhou, Tian Lan, Guru Venkataramani, and Peng Wei. Mac-po: Multi-agent experience replay via collective priority optimization. In *Proceedings of the 2023 International Conference* on Autonomous Agents and Multiagent Systems, pages 466–475, 2023.
- [33] Youssef Mroueh, Stephen Voinea, and Tomaso A Poggio. Learning with group invariant features: A kernel perspective. *Advances in neural information processing systems*, 28, 2015.
- [34] Qi Qi, Youzhi Luo, Zhao Xu, Shuiwang Ji, and Tianbao Yang. Stochastic optimization of areas under precision-recall curves with provable convergence. Advances in Neural Information Processing Systems, 34:1752–1765, 2021.
- [35] Sashank Reddi, Zachary Charles, Manzil Zaheer, Zachary Garrett, Keith Rush, Jakub Konečný, Sanjiv Kumar, and H Brendan McMahan. Adaptive federated optimization. arXiv preprint arXiv:2003.00295, 2020.
- [36] Rahul Singh, Maneesh Sahani, and Arthur Gretton. Kernel instrumental variable regression. *Advances in Neural Information Processing Systems*, 32, 2019.
- [37] Oriol Vinyals, Charles Blundell, Timothy Lillicrap, Daan Wierstra, et al. Matching networks for one shot learning. *Advances in neural information processing systems*, 29, 2016.
- [38] Bokun Wang and Tianbao Yang. Finite-sum coupled compositional stochastic optimization: Theory and applications. In International Conference on Machine Learning, ICML 2022, 17-23 July 2022, Baltimore, Maryland, USA, volume 162 of Proceedings of Machine Learning Research, pages 23292–23317. PMLR, 2022.

- [39] Bokun Wang, Zhuoning Yuan, Yiming Ying, and Tianbao Yang. Memory-based optimization methods for model-agnostic meta-learning. arXiv preprint arXiv:2106.04911, 2021.
- [40] Guanghui Wang, Ming Yang, Lijun Zhang, and Tianbao Yang. Momentum accelerates the convergence of stochastic aupre maximization. In *International Conference on Artificial Intelligence and Statistics*, pages 3753–3771. PMLR, 2022.
- [41] Jialu Wang, Ping Li, and Feifang Hu. A/b testing in network data with covariate-adaptive randomization. 2023.
- [42] Mengdi Wang, Ji Liu, and Ethan Fang. Accelerating stochastic composition optimization. Advances in Neural Information Processing Systems, 29, 2016.
- [43] Mengdi Wang, Ethan X Fang, and Han Liu. Stochastic compositional gradient descent: algorithms for minimizing compositions of expected-value functions. *Mathematical Programming*, 161(1):419–449, 2017.
- [44] Xidong Wu, Feihu Huang, and Heng Huang. Fast stochastic recursive momentum methods for imbalanced data mining. In 2022 IEEE International Conference on Data Mining (ICDM), pages 578–587. IEEE, 2022.
- [45] Xidong Wu, Zhengmian Hu, Jian Pei, and Heng Huang. Serverless federated auprc optimization for multi-party collaborative imbalanced data mining. In *Proceedings of the 29th ACM SIGKDD Conference* on Knowledge Discovery and Data Mining, pages 2648–2659, 2023.
- [46] Xidong Wu, Feihu Huang, Zhengmian Hu, and Heng Huang. Faster adaptive federated learning. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 37, pages 10379–10387, 2023.
- [47] Xidong Wu, Jianhui Sun, Zhengmian Hu, Aidong Zhang, and Heng Huang. Solving a class of nonconvex minimax optimization in federated learning. Advances in Neural Information Processing Systems (NeurIPS), 2023.
- [48] Huizhuo Yuan and Wenqing Hu. Stochastic recursive momentum method for non-convex compositional optimization. *arXiv preprint arXiv:2006.01688*, 2020.
- [49] Zhuoning Yuan, Zhishuai Guo, Yi Xu, Yiming Ying, and Tianbao Yang. Federated deep auc maximization for hetergeneous data with a constant communication complexity. In *International Conference on Machine Learning*, pages 12219–12229. PMLR, 2021.
- [50] Junyu Zhang and Lin Xiao. A composite randomized incremental gradient method. In *International Conference on Machine Learning*, pages 7454–7462. PMLR, 2019.

A Supplementary material

 $\begin{array}{lll} s_t \mbox{ denotes the } s_t &= \lfloor t/q \rfloor. \mbox{ We define } g^n(x,\xi^n) &= \mathbb{E}_{\eta^n \mid \xi^n} g^n_{\eta^n}(x,\xi^n) \mbox{ and } \hat{g}^n(x,\xi) &= \frac{1}{m} \sum_{j=1}^m g^n(x,\xi;\eta_j), \hat{F}^n(x;\xi^n,\{\eta^n_j\}_{j=1}^m) = f^n_{\xi^n}(\hat{g}^n(x,\xi^n)). \end{array}$

A.1 Basic Lemma

For convenience, in the subsequent analysis, s_t denotes the $s_t = \lfloor t/q \rfloor$ and $s_t \in [\lfloor T/q \rfloor]$. $g^n(x, \xi^n) = \mathbb{E}_{\eta^n \mid \xi^n} g^n_{\eta^n}(x, \xi^n)$ and $\hat{g}^n(x, \xi^n) = \frac{1}{m} \sum_{j=1}^m g^n(x, \xi^n; \eta_j), \hat{F}^n(x; \xi^n, \{\eta^n_j\}_{j=1}^m) = f^n_{\xi^n}(\hat{g}^n(x, \xi^n)).$

$$\nabla F^{n}(x) = \nabla \mathbb{E}_{\xi^{n}} \left[f^{n}_{\xi^{n}}(g^{n}(x,\xi^{n})) \right] = \mathbb{E}_{\xi^{n}} \left[\nabla \left(f^{n}_{\xi^{n}}(g^{n}(x,\xi^{n})) \right) \right]$$
$$= \mathbb{E}_{\xi^{n}} \left[\nabla g^{n}(x,\xi^{n})^{\top} \cdot \nabla_{g} f^{n}_{\xi^{n}}(g^{n}(x,\xi^{n})) \right]$$
(7)

$$\hat{F}^{n}(x) = \mathbb{E}\hat{F}^{n}(x;\xi^{n},\{\eta_{j}^{n}\}_{j=1}^{m}) = \mathbb{E}_{\xi^{n}}\mathbb{E}_{\eta^{n}|\xi^{n}}\left[f_{\xi^{n}}^{n}(\hat{g}^{n}(x,\xi^{n}))\right],\nabla\hat{F}^{n}(x)$$
$$= \mathbb{E}\nabla\hat{F}^{n}(x;\xi^{n},\{\eta_{j}^{n}\}_{j=1}^{m})$$

$$F(x) = \frac{1}{N} \sum_{n=1}^{N} F^n(x), \hat{F}(x) = \frac{1}{N} \sum_{n=1}^{N} \hat{F}^n(x)$$
(8)

Lemma A.1. [27] For $x_t^n \in \mathbb{R}^d$, $n \in [N]$ and $\bar{x}_t \in \mathbb{R}^d$, we have

$$\sum_{n=1}^{N} \|x_t^n - \bar{x}_t\|^2 \le \sum_{n=1}^{N} \|x_t^n\|^2$$
(9)

Lemma A.2. Under Assumption 3.1 3.2 and 3.3, in the n-th device, for a sample ξ^n and m i.i.d. samples $\{\eta_j^n\}_{j=1}^m$ from the conditional distribution $P(\eta^n | \xi^n)$, and $\forall x, x_1, x_2 \in \mathcal{X}$ that is independent of ξ^n and $\{\eta_j^n\}_{j=1}^m$, we have (a) (Lemma 2.2 in [19])

$$\|\mathbb{E}\nabla\widehat{F}^{n}(x;\xi^{n},\{\eta_{j}^{n}\}_{j=1}^{m}) - \nabla F^{n}(x)\|^{2} \leq \frac{L_{g}^{2}S_{f}^{2}\sigma_{g}^{2}}{m}$$
(10)

(b) (Proposition B.1 in [20] $F^n(x)$ and $\hat{F}^n(x)$ are S_F -Lipschitz smooth where $S_F = S_g L_f + S_f L_g^2$ and we also have

$$\mathbb{E}\left\|\nabla\hat{F}^{n}\left(x_{1};\xi,\{\eta_{j}\}_{j=1}^{m}\right) - \nabla\hat{F}^{n}\left(x_{2};\xi,\{\eta_{j}\}_{j=1}^{m}\right)\right\| \leq S_{F}\|x_{1} - x_{2}\|$$
(11)

(c) (Proposition B.1 in [20])

$$\mathbb{E}_{\xi^n} \left\| \nabla f_{\xi^n}^n(y) - \nabla \mathbb{E} f_{\xi^n}^n(y) \right\|_2^2 \le L_f^2 \quad \mathbb{E}_{\xi^n \mid \eta^n} \left\| \nabla g_{\eta}^n(x,\xi^n) - \nabla \mathbb{E} g_{\eta}^n(x,\xi^n) \right\|_2^2 \le L_g^2$$
$$\mathbb{E}_{\xi^n} \mathbb{E}_{\xi^n \mid \eta^n} \left\| \nabla \left(f_{\xi^n}^n(\hat{g}^n(x,\xi^n)) \right) - \nabla \hat{F}^n(x) \right\|^2 \le L_f^2 L_g^2 \quad \left\| \nabla \hat{F}^n(x) \right\|^2 \le L_f^2 L_g^2$$

(d) (Bounded Heterogeneity)

$$\left\|\nabla \hat{F}^n(x) - \nabla \hat{F}^k(x)\right\|^2 \le 4L_f^2 L_g^2 \tag{12}$$

Proof. (e) Based on the (d), we know $\nabla \hat{F}^n(x)$ has bounded gradient, therefore,

$$\left\|\nabla \hat{F}^{n}(x) - \nabla \hat{F}^{k}(x)\right\|^{2} \le 2\|\nabla \hat{F}^{n}(x)\|^{2} + 2\|\nabla \hat{F}^{k}(x)\|^{2} \le 4L_{f}^{2}L_{g}^{2}$$

Lemma A.3. For $n \in [N]$, we have

$$\mathbb{E} \|\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n}; \xi_{t,i}^{n}, \mathcal{B}_{t,i}^{n}) - \nabla \hat{F}^{n}(x_{t}^{n})\|^{2} \leq \frac{L_{f}^{2} L_{g}^{2}}{b}$$
$$\sum_{n=1}^{N} \mathbb{E} \|\nabla \hat{F}^{n}(x_{t}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \nabla \hat{F}^{k}(x_{t}^{k})\|^{2} \leq 6S_{F}^{2} \sum_{n=1}^{N} \mathbb{E} \|x_{t}^{n} - \bar{x}_{t}\|^{2} + 12NL_{f}^{2} L_{g}^{2}$$

Proof. 1) we have

$$\begin{split} \mathbb{E} \| \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2} &= \frac{1}{b^{2}} \mathbb{E} \| \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2} \\ &= \frac{1}{b^{2}} \sum_{i=1}^{b} \mathbb{E} \| \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2} \leq \frac{L_{f}^{2} L_{g}^{2}}{b} \end{split}$$

where the second equality is due to $\mathbb{E}[\nabla \hat{F}^n(x_t^n; \xi_{t,i}^n, \mathcal{B}_{t,i}^n) - \nabla \hat{F}^n(x_t^n)] = 0$ and the last inequality follows Lemma A.2 (c).

2)

$$\sum_{n=1}^{N} \mathbb{E} \|\nabla \hat{F}^{n}(x_{t}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \nabla \hat{F}^{k}(x_{t}^{k})\|^{2}$$

$$\leq 3 \sum_{n=1}^{N} \mathbb{E} \left[\|\nabla \hat{F}^{n}(x_{t}^{n}) - \nabla \hat{F}^{n}(\bar{x}_{t})\|^{2} + \|\nabla \hat{F}(\bar{x}_{t}) - \frac{1}{N} \sum_{k=1}^{N} \nabla \hat{F}(x_{t}^{k})\|^{2} + \|\nabla \hat{F}^{n}(\bar{x}_{t}) - \nabla \hat{F}(\bar{x}_{t})\|^{2} \right]$$

$$\leq 6 S_{F}^{2} \sum_{n=1}^{N} \mathbb{E} \|x_{t}^{n} - \bar{x}_{t}\|^{2} + 3 \sum_{n=1}^{N} \mathbb{E} \|\nabla \hat{F}^{n}(\bar{x}_{t}) - \nabla \hat{F}(\bar{x}_{t})\|^{2}$$

$$\leq 6 S_{F}^{2} \sum_{n=1}^{N} \mathbb{E} \|x_{t}^{n} - \bar{x}_{t}\|^{2} + 12 N L_{f}^{2} L_{g}^{2}$$

$$(13)$$

where the second inequality is due to Assumption A.2 (b) and the last inequality is due to Assumption A.2 (d). $\hfill \Box$

B Proof of FCSG Algorithm

B.1 Proofs of the Intermediate Lemmas

Lemma B.1. Assume the sequence $\{\bar{x}_t\}_{t=0}^{T-1}$ is generated from FCSG in Algorithm 1, if $\alpha \leq \frac{1}{6S_Fq}$, we have

$$\sum_{t=0}^{T-1} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \left\| x_t^n - \bar{x}_t \right\|^2 \le 39(q-1)q\alpha^2 L_f^2 L_g^2 T \tag{14}$$

Proof. (1) if $t = s_t q$, we have

$$\sum_{n=1}^{N} \left\| x_{s_t q}^n - \bar{x}_{s_t q} \right\|^2 = 0$$
(15)

(2) if $t > s_t q$, we have

$$x_t^n = x_{s_tq}^n - \sum_{s=s_tq+1}^t \alpha u_s^n \quad \bar{x}_t = \bar{x}_{s_tq} - \sum_{s=s_tq+1}^t \alpha \bar{u}_s$$
(16)

It implies that

$$\frac{1}{N}\sum_{n=1}^{N} \mathbb{E} \|x_t^n - \bar{x}_t\|^2 = \frac{1}{N}\sum_{n=1}^{N} \mathbb{E} \left\| x_{s_tq}^n - \bar{x}_{s_tq} - \left(\sum_{s=s_tq+1}^t \alpha u_s^n - \sum_{s=s_tq+1}^t \alpha \bar{u}_s\right) \right\|^2$$
$$= \frac{1}{N}\sum_{n=1}^{N} \mathbb{E} \|\sum_{s=s_tq+1}^t \alpha (u_s^n - \bar{u}_s)\|^2$$
$$\leq \frac{(q-1)\alpha^2}{N}\sum_{s=s_tq+1}^t \sum_{n=1}^{N} \mathbb{E} \|u_s^n - \bar{u}_s\|^2$$
(17)

Furthermore, based on the definition of u_t^n , we have

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\|x_{t}^{n}-\bar{x}_{t}\|^{2} \\
\leq \frac{(q-1)\alpha^{2}}{N}\sum_{s=s_{t}q}^{t-1}\sum_{n=1}^{N}\mathbb{E}\left\|\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n})-\frac{1}{N}\sum_{k=1}^{N}\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{k}(x_{t}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k})\right\|^{2} \\
\stackrel{(a)}{\leq} \frac{2(q-1)\alpha^{2}}{N}\sum_{s=s_{t}q}^{t-1}\sum_{n=1}^{N}\mathbb{E}\left[\left\|\left[\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n})-\nabla\hat{F}^{n}(x_{t}^{n})\right]\right. \\
\left.-\frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{k}(x_{t}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k})-\nabla\hat{F}^{k}(x_{t}^{k})\right]\right\|^{2} +\left\|\left[\nabla\hat{F}^{n}(x_{t}^{n})-\frac{1}{N}\sum_{k=1}^{N}\nabla\hat{F}^{k}(x_{t}^{k})\right]\right\|^{2}\right] \\
\stackrel{(b)}{\leq} \frac{2(q-1)\alpha^{2}}{N}\sum_{s=s_{t}q}^{t-1}\left[\sum_{n=1}^{N}\mathbb{E}\left\|\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n})-\nabla\hat{F}^{n}(x_{t})\right\|^{2} +\sum_{n=1}^{N}\mathbb{E}\|\hat{F}^{n}(x_{t}^{n})-\hat{F}(x_{t})\|^{2}\right] \\
\stackrel{(c)}{\leq} \sum_{s=s_{t}q}^{t-1}\left[26(q-1)\alpha^{2}L_{f}^{2}L_{g}^{2}+\frac{12(q-1)\alpha^{2}S_{F}^{2}}{N}\sum_{n=1}^{N}\mathbb{E}\|x_{t}^{n}-\bar{x}_{t}\|^{2}\right] \\
\leq 26(q-1)q\alpha^{2}L_{f}^{2}L_{g}^{2}+\frac{12(q-1)q\alpha^{2}S_{F}^{2}}{N}\sum_{n=1}^{N}\mathbb{E}\|x_{t}^{n}-\bar{x}_{t}\|^{2}\right] \tag{18}$$

where (a) holds by $||a + b||^2 \le 2||a||^2 + 2||b||^2$; the (b) holds due to Lemma A.1; the (c) holds by Lemma A.3. Summing both sides from $s = s_t q$ to \bar{s} where $\bar{s} = [s_t q, (s_t + 1)q)$, we get

$$\sum_{s=s_tq}^{\bar{s}} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \left\| x_s^n - \bar{x}_s \right\|^2 \le \sum_{s=s_tq}^{\bar{s}} [26(q-1)q\alpha^2 L_f^2 L_g^2 + \frac{12(q-1)q\alpha^2 S_F^2}{N} \sum_{n=1}^{N} \mathbb{E} \| x_s^n - \bar{x}_s \|^2]$$

Rearranging the terms, we get

$$\left(1 - 12S_F^2 q^2 \alpha^2\right) \sum_{s=s_t q}^{\bar{s}} \frac{1}{N} \sum_{n=1}^N \mathbb{E} \left\| x_s^n - \bar{x}_s \right\|^2 \le \sum_{s=s_t q}^{\bar{s}} 26(q-1)q\alpha^2 L_f^2 L_g^2 \tag{19}$$

Finally, using the fact that $\alpha q \leq \frac{1}{6S_F}$ we have $1 - 12S_F^2 q^2 \alpha^2 \geq 2/3$. Multiplying, both sides by 3/2 we get

$$\sum_{s=s_tq}^{\bar{s}} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \|x_t^n - \bar{x}_t\|^2 \le \sum_{s=s_tq}^{\bar{s}} 39(q-1)q\alpha^2 L_f^2 L_g^2$$
(20)

Finally, when we consider the sum over t from 0 to T - 1, we get

$$\sum_{t=0}^{T-1} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \left\| x_t^n - \bar{x}_t \right\|^2 \le 39(q-1)q\alpha^2 L_f^2 L_g^2 T$$
(21)

Lemma B.2. Suppose the sequence $\{x_t\}_{t=0}^T$ be generated from FCSG in Algorithms 1. We have

$$\mathbb{E}F(\bar{x}_{t+1}) \le \mathbb{E}F(\bar{x}_t) - \frac{\alpha}{2} \mathbb{E}\|\nabla F(\bar{x}_t)\|^2 + \frac{\alpha S_F^2}{N} \sum_{n=1}^N \|x_t^n - \bar{x}_t\|^2 + \frac{\alpha L_g^2 S_f^2 \sigma_g^2}{m} + \frac{\alpha^2 S_F L_f^2 L_g^2}{N}$$

Proof.

$$\begin{split} \mathbb{E}F(\bar{x}_{t+1}) \\ & \stackrel{(a)}{\leq} \mathbb{E}F(\bar{x}_{t}) + \mathbb{E}\langle \nabla F(\bar{x}_{t}), \bar{x}_{t+1} - \bar{x}_{t} \rangle + \frac{S_{F}}{2} \mathbb{E} \| \bar{x}_{t+1} - \bar{x}_{t} \|^{2} \\ & \stackrel{(b)}{\leq} \mathbb{E}F(\bar{x}_{t}) - \alpha \mathbb{E}\langle \nabla F(\bar{x}_{t}), \bar{u}_{t+1} \rangle + \frac{\alpha^{2} S_{F}}{2} \mathbb{E} \| \bar{u}_{t+1} \|^{2} \\ & \stackrel{(c)}{\equiv} \mathbb{E}F(\bar{x}_{t}) - (\frac{\alpha}{2} - \alpha^{2} S_{F}) \| \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2} - \frac{\alpha}{2} \| \nabla F(\bar{x}_{t}) \|^{2} + \frac{\alpha}{2} \mathbb{E} \| \nabla F(\bar{x}_{t}) - \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2} \\ & + \alpha^{2} S_{F} \mathbb{E} \| \bar{u}_{t+1} - \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2} \\ & \stackrel{(d)}{\leq} \mathbb{E}F(\bar{x}_{t}) - \frac{\alpha}{2} \mathbb{E} \| \nabla F(\bar{x}_{t}) \|^{2} + \frac{\alpha}{2} \mathbb{E} \| \nabla F(\bar{x}_{t}) - \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2} + \alpha^{2} S_{F} \mathbb{E} \| \bar{u}_{t+1} - \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2} \\ & \leq \mathbb{E}F(\bar{x}_{t}) - \frac{\alpha}{2} \mathbb{E} \| \nabla F(\bar{x}_{t}) \|^{2} + \alpha \mathbb{E} \| \nabla F(\bar{x}_{t}) - \frac{1}{N} \sum_{n=1}^{N} \nabla F^{n}(x_{t}^{n}) \|^{2} \\ & \leq \mathbb{E}F(\bar{x}_{t}) - \frac{\alpha}{2} \mathbb{E} \| \nabla F(\bar{x}_{t}) \|^{2} + \alpha \mathbb{E} \| \nabla F(\bar{x}_{t}) - \frac{1}{N} \sum_{n=1}^{N} \nabla F^{n}(x_{t}^{n}) \|^{2} \\ & + \alpha \mathbb{E} \| \frac{1}{N} \sum_{n=1}^{N} \nabla F^{n}(x_{t}^{n}) - \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2} + \alpha^{2} S_{F} \mathbb{E} \| \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) - \bar{u}_{t+1} \|^{2} \end{aligned} \tag{22}$$

where inequality (a) holds by the smoothness of F(x); equality (b) follows from update step in Step 9 of Algorithm 1; (c) uses the fact that $\langle a, b \rangle = \frac{1}{2}[\|a\|^2 + \|a\|^2 - \|a - b\|^2]$, $\|a + b\|^2 \le 2\|a\|^2 + 2\|b\|^2$, and $\mathbb{E}[\bar{u}_{t+1}] = \nabla \hat{F}(x_t)$; (d) results from that $\alpha S_F \le \frac{1}{2}$. Taking expectation on both sides and considering the last third term

$$\mathbb{E} \|\nabla F(\bar{x}_t) - \frac{1}{N} \sum_{n=1}^N \nabla F^n(x_t^n) \|^2 \le \frac{1}{N} \sum_{n=1}^N \mathbb{E} \|\nabla F^n(\bar{x}_t) - \nabla F^n(x_t^n) \|^2 \le \frac{S_F^2}{N} \sum_{n=1}^N \|x_t^n - \bar{x}_t\|^2$$
(23)

Considering the last second term and using Lemma A.2 (a), we have

$$\mathbb{E} \| \frac{1}{N} \sum_{n=1}^{N} \nabla F^{n}(x_{t}^{n}) - \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2}$$

$$\leq \frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \| \nabla F^{n}(x_{t}^{n}) - \nabla \hat{F}^{n}(x_{t}^{n}) \|^{2} \leq \frac{L_{g}^{2} S_{f}^{2} \sigma_{g}^{2}}{m}$$
(24)

For the last term, given that $\mathbb{E}\bar{u}_{t+1} = \frac{1}{N}\sum_{n=1}^{N}\nabla\hat{F}^n(x_t^n)$, we have

$$\mathbb{E} \| \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) - \bar{u}_{t+1} \|^{2} \le \frac{L_{f}^{2} L_{g}^{2}}{N}$$
(25)

Therefore, we obtain the final result.

B.2 Proof of Theorem 4.1

Based on previous lemmas, we start to prove the convergence of Theorem 4.1.

Proof. Taking the telescoping sum of B.2 over t from 0 to T - 1,

$$\frac{1}{T} \sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \leq \frac{2[F(\bar{x}_0) - F(\bar{x}_T)]}{\alpha T} + \frac{2S_F^2}{TN} \sum_{t=0}^{T-1} \sum_{n=1}^N \|x_t^n - \bar{x}_t\|^2 + \frac{2L_g^2 S_f^2 \sigma_g^2}{m} + \frac{2\alpha S_F L_f^2 L_g^2}{N} \leq \frac{2[F(\bar{x}_0) - F(\bar{x}_T)]}{\alpha T} + \frac{2L_g^2 S_f^2 \sigma_g^2}{m} + \frac{2\alpha S_F L_f^2 L_g^2}{N} + 78(q-1)q\alpha^2 L_f^2 L_g^2 S_F^2.$$
(26)

where the second inequality holds due to Lemma B.1. Furthermore, we choose $\alpha = \frac{1}{6S_F} \sqrt{\frac{N}{T}}$ and $q = (T/N^3)^{1/4}$, we have

$$\frac{1}{T}\sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \le \frac{12S_F[F(\bar{x}_0) - F(\bar{x}_*)]}{(NT)^{1/2}} + \frac{2L_g^2 S_f^2 \sigma_g^2}{m} + \frac{L_f^2 L_g^2}{6(NT)^{1/2}} + \frac{19L_f^2 L_g^2}{9(NT)^{1/2}} \tag{27}$$

To let the right hand side less than ε^2 when $m = \varepsilon^{-2}$, we get $T = O(N^{-1}\varepsilon^{-4})$ and $\frac{T}{q} = (NT)^{3/4} = \varepsilon^{-3}$.

C Proof of FCSG-M Algorithm

C.1 Proofs of the Intermediate Lemmas

Lemma C.1. Suppose the sequence $\{x_t\}_{t=0}^T$ be generated from FCSG-M in Algorithms 1. We have

$$\mathbb{E}F(\bar{x}_{t+1}) \leq \mathbb{E}F(\bar{x}_t) - (\frac{\alpha}{2} - \frac{\alpha^2 S_F}{2}) \mathbb{E} \|\bar{u}_{t+1}\|^2 - \frac{\alpha}{2} \mathbb{E} \|\nabla F(\bar{x}_t)\|^2 + \frac{4\alpha S_F^2}{N} \sum_{n=1}^N \mathbb{E} \|x_t^n - \bar{x}_t\|^2 + \frac{2\alpha L_g^2 S_f^2 \sigma_g^2}{m} + 2\alpha \mathbb{E} \|\nabla \hat{F}(\bar{x}_t) - \bar{u}_{t+1}\|^2$$
(28)

Proof.

$$\begin{split} F(\bar{x}_{t+1}) \\ &\leq F(\bar{x}_{t}) + \langle \nabla F(\bar{x}_{t}), \bar{x}_{t+1} - \bar{x}_{t} \rangle + \frac{S_{F}}{2} \|\bar{x}_{t+1} - \bar{x}_{t}\|^{2} \\ &\leq F(\bar{x}_{t}) - \alpha \langle \nabla F(\bar{x}_{t}), \bar{u}_{t+1} \rangle + \frac{\alpha^{2} S_{F}}{2} \|\bar{u}_{t+1}\|^{2} \\ &\leq F(\bar{x}_{t}) - (\frac{\alpha}{2} - \frac{\alpha^{2} S_{F}}{2}) \|\bar{u}_{t+1}\|^{2} - \frac{\alpha}{2} \|\nabla F(\bar{x}_{t})\|^{2} + \frac{\alpha}{2} \|\nabla F(\bar{x}_{t}) - \bar{u}_{t+1}\|^{2} \\ &\leq F(\bar{x}_{t}) - (\frac{\alpha}{2} - \frac{\alpha^{2} S_{F}}{2}) \|\bar{u}_{t+1}\|^{2} - \frac{\alpha}{2} \|\nabla F(\bar{x}_{t})\|^{2} + \frac{4\alpha}{2} \|\nabla F(\bar{x}_{t}) - \frac{1}{N} \sum_{n=1}^{N} \nabla F^{n}(x_{t}^{n})\|^{2} \\ &+ \frac{4\alpha}{2} \|\frac{1}{N} \sum_{n=1}^{N} [\nabla F^{n}(x_{t}^{n}) - \nabla \hat{F}^{n}(x_{t}^{n})] \|^{2} + \frac{4\alpha}{2} \|\frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) - \nabla \hat{F}(\bar{x}_{t})\|^{2} \\ &+ \frac{4\alpha}{2} \|\nabla \hat{F}(\bar{x}_{t}) - \bar{u}_{t+1}\|^{2} \end{split}$$

where inequality (a) holds by the smoothness of F(x); equality (b) follows from update step in Step 9 of FCSG-M in Algorithm 1; (c) uses the fact that $\langle a, b \rangle = \frac{1}{2}[\|a\|^2 + \|a\|^2 - \|a - b\|^2]$. Taking

expectation on both sides and give the fact that

$$\begin{split} \mathbb{E} \|\nabla F(\bar{x}_t) - \frac{1}{N} \sum_{n=1}^N \nabla F^n(x_t^n) \|^2 &\leq \frac{1}{N} \sum_{n=1}^N \mathbb{E} \|\nabla F^n(\bar{x}_t) - \nabla F^n(x_t^n) \|^2 \leq \frac{S_F^2}{N} \sum_{n=1}^N \mathbb{E} \|x_t^n - \bar{x}_t\|^2 \\ \mathbb{E} \|\frac{1}{N} \sum_{n=1}^N \nabla F^n(x_t) - \frac{1}{N} \sum_{n=1}^N \nabla \hat{F}^n(x_t) \|^2 &\leq \frac{1}{N} \sum_{n=1}^N \mathbb{E} \|\nabla F^n(x_t) - \nabla \hat{F}^n(x_t) \|^2 \overset{(a)}{\leq} \frac{L_g^2 S_f^2 \sigma_g^2}{m} \\ \mathbb{E} \|\frac{1}{N} \sum_{n=1}^N \nabla \hat{F}^n(x_t^n) - \nabla \hat{F}(\bar{x}_t) \|^2 &\leq \frac{S_F^2}{N} \sum_{n=1}^N \mathbb{E} \|x_t^n - \bar{x}_t\|^2 \end{split}$$

where (a) holds due to Lemma A.2 (a). Therefore, taking the expectation on both sides, we obtain the final results. $\hfill \Box$

Lemma C.2. Suppose that the sequence $\{u_t\}_{t=1}^T$ be generated from FCSG-M in Algorithm 1, we have

$$\begin{aligned} &\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left\| \bar{u}_{t+1} - \nabla \hat{F} \left(\bar{x}_t \right) \right\|^2 \\ \leq &\frac{2}{\beta T} \mathbb{E} \left\| \bar{u}_1 - \nabla \hat{F} \left(\bar{x}_0 \right) \right\|^2 + \frac{4\alpha^2 S_F^2}{\beta^2 T} \sum_{t=0}^{T-1} \mathbb{E} \left\| \bar{u}_{t+1} \right\|^2 + \frac{2\beta L_f^2 L_g^2}{N} + \frac{2S_F^2}{NT} \sum_{t=0}^{T-1} \sum_{n=1}^N \mathbb{E} \left\| x_t^n - \bar{x}_t \right\|^2 \end{aligned}$$

Proof. Recall that $\bar{u}_{t+1} = \frac{1}{N} \sum_{n=1}^{N} \left[\frac{\beta}{b} \sum_{i=1}^{b} \nabla \hat{F}^n(x_t^n; \xi_{t,i}^n, \mathcal{B}_{t,i}^n) + (1-\beta)u_t^n\right],$

$$\begin{split} & \mathbb{E} \left\| \bar{u}_{t+1} - \nabla \hat{F}\left(\bar{x}_{t}\right) \right\|^{2} = \mathbb{E} \left\| \frac{\beta}{N} \sum_{n=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) + (1-\beta)\bar{u}_{t} - \nabla \hat{F}\left(\bar{x}_{t}\right) \right\|^{2} \\ & = \mathbb{E} \| \nabla \hat{F}\left(\bar{x}_{t}\right) - (1-\beta)\bar{u}_{t} - \beta \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}\left(\mathbf{x}_{t}^{n}\right) - \beta \frac{1}{N} \sum_{n=1}^{N} [\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \nabla \hat{F}^{n}\left(\mathbf{x}_{t}^{n}\right)] \|^{2} \\ & = \mathbb{E} \left\| (1-\beta)(\nabla \hat{F}(\bar{x}_{t}) - \bar{u}_{t}) + \beta \left(\nabla \hat{F}\left(\bar{x}_{t}\right) - \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}^{n}) \right) \right\|^{2} \\ & + \beta^{2} \mathbb{E} \left\| \frac{1}{N} \sum_{n=1}^{N} [\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \nabla \hat{F}^{n}\left(\mathbf{x}_{t}^{n}\right)] \right\|^{2} \\ & + \beta^{2} \mathbb{E} \left\| \frac{1}{N} \sum_{n=1}^{N} [\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \nabla \hat{F}^{n}\left(\mathbf{x}_{t}^{n}\right)] \right\|^{2} \\ & + \beta^{2} \frac{L_{f}^{2} L_{g}^{2}}{N} \end{split}$$

Here, (a) holds due to the definition of $\nabla \hat{F}^n(x_t^n)$. (b) holds due to Young's inequality and A.2 (c). We choose $c_1 = \frac{\beta}{1-\beta}$, and $(1-\beta)(1+c_1) = 1$ and $(1+\frac{1}{c_1})\beta = 1$. Therefore, we have

where in inequality (a) we choose $c_2 = \frac{\beta}{2}$. Then, $(1 - \beta) \left(1 + \frac{\beta}{2}\right) \le 1 - \frac{\beta}{2}$, and $(1 - \beta) \left(1 + \frac{2}{\beta}\right) \le \frac{2}{\beta}$. Then we have

$$\mathbb{E} \left\| \bar{u}_{t+1} - \nabla \hat{F}(\bar{x}_t) \right\|^2 - \mathbb{E} \left\| \bar{u}_t - \nabla \hat{F}(\bar{x}_{t-1}) \right\|^2 \\ \leq -\frac{\beta}{2} \mathbb{E} \left\| \bar{u}_t - \nabla \hat{F}(\bar{x}_{t-1}) \right\|^2 + \frac{2\alpha^2 S_F^2}{\beta} \mathbb{E} \left\| \bar{u}_t \right\|^2 + \frac{\beta^2 L_f^2 L_g^2}{N} + \frac{\beta S_F^2}{N} \sum_{n=1}^N \mathbb{E} \left\| x_t^n - \bar{x}_t \right\|^2$$

By rearranging the terms and summing the t from 0 to T - 1, one can get the final result.

Lemma C.3. Assume $\alpha \leq \frac{1}{6qS_F}$ and the sequence $\{\bar{x}_t\}_{t=0}^{T-1}$ be generated from FCSG-M in Algorithm 1, the consensus error $\mathbb{E} \|x_t^n - \bar{x}_t\|^2$ satisfies

$$\sum_{t=0}^{T-1} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \left\| x_t^n - \bar{x}_t \right\|^2 \le \frac{6q(q-1)\alpha^2 T}{5} [L_f^2 L_g^2 (1+\frac{1}{N}) + 12L_f^2 L_g^2 (1+\frac{1}{N}) + 12$$

$$\begin{split} &\frac{1}{N}\sum_{n=1}^{N}\|u_{t+1}^{n}-\bar{u}_{t+1}\|^{2} \\ &=\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left\|\left(1-\beta\right)\left(u_{t}^{n}-\bar{u}_{t}\right)+\beta\left[\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n})-\frac{1}{N}\sum_{k=1}^{N}\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{k}(x_{t}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k})\right)\right\|^{2} \\ &\leq \frac{(1+c_{3})\left(1-\beta\right)^{2}}{N}\sum_{n=1}^{N}\mathbb{E}\left\|u_{t}^{n}-\bar{u}_{t}\right\|^{2} \\ &+\left(1+\frac{1}{c_{3}}\right)\frac{\beta^{2}}{N}\sum_{n=1}^{N}\mathbb{E}\left\|\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n})-\frac{1}{N}\sum_{k=1}^{N}\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{k}(x_{t}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k})\right\|^{2} \\ &\leq \frac{(1-\beta)}{N}\sum_{n=1}^{N}\mathbb{E}\left\|u_{t}^{n}-\bar{u}_{t}\right\|^{2}+\frac{\beta}{N}\sum_{n=1}^{N}\mathbb{E}\left\|\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n})-\nabla\hat{F}^{n}(x_{t}^{n})+\nabla\hat{F}^{n}(x_{t}^{n})\right|^{2} \\ &-\nabla\hat{F}^{n}\left(\bar{x}_{t}\right)+\nabla\hat{F}^{n}\left(\bar{x}_{t}\right)-\frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{k}(x_{t}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k})-\nabla\hat{F}^{k}\left(x_{t}^{k}\right)\right] \\ &+\frac{1}{N}\sum_{k=1}^{n}\left[\nabla\hat{F}^{k}\left(\bar{x}_{t}\right)-\nabla\hat{F}^{k}\left(\bar{x}_{t}\right)\right]\right\|^{2} \\ &\leq \frac{(1-\beta)}{N}\sum_{n=1}^{N}\mathbb{E}\left\|u_{t}^{n}-\bar{u}_{t}\right\|^{2}+\frac{\beta}{N}\sum_{n=1}^{N}\mathbb{E}\left[\left\|\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n})-\nabla\hat{F}^{n}\left(x_{t}^{n}\right)\right\|^{2} \\ &+\left\|\frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{b}\sum_{i=1}^{b}\nabla\hat{F}^{k}(x_{t}^{k};\xi_{k,i}^{k},\mathcal{B}_{t,i}^{k})-\nabla\hat{F}^{k}\left(x_{t}^{k}\right)\right]\right\|^{2} \\ &+\left\|\frac{1}{N}\sum_{k=1}^{N}\left[\nabla\hat{F}^{k}\left(x_{t}^{k}\right)-\nabla\hat{F}^{k}\left(\bar{x}_{t}\right)\right]-\nabla\hat{F}^{k}\left(\bar{x}_{t}^{k}\right)\right]\right\|^{2} \\ &+\left\|\frac{1}{N}\sum_{k=1}^{N}\left[\nabla\hat{F}^{k}\left(x_{t}^{k}\right)-\nabla\hat{F}^{k}\left(\bar{x}_{t}\right)\right]-\nabla\hat{F}^{k}\left(\bar{x}_{t}^{k}\right)\right\|^{2} \\ &\leq \frac{(1-\beta)}{N}\sum_{n=1}^{N}\mathbb{E}\left\|u_{t}^{n}-\bar{u}_{t}\right\|^{2} + \frac{\beta}{N}\sum_{n=1}^{N}\left[L^{2}_{i}L^{2}_{i}g^{2}+\frac{L^{2}_{i}L^{2}_{i}g}}{N}+3\mathbb{E}\left\|\nabla\hat{F}^{n}\left(x_{t}^{n}\right)-\nabla\hat{F}^{n}\left(\bar{x}_{t}\right)\right\|^{2} \\ &+3\mathbb{E}\left\|\nabla\hat{F}^{n}\left(\bar{x}_{t}\right)-\nabla\hat{F}\left(\bar{x}_{t}\right)\right\|^{2} + 3\mathbb{E}\left\|\nabla\hat{F}^{n}\left(x_{t}^{n}\right)-\nabla\hat{F}^{n}\left(\bar{x}_{t}\right)\right\|^{2} \\ &\leq \frac{(1-\beta)}{N}\sum_{n=1}^{N}\mathbb{E}\left\|u_{t}^{n}-\bar{u}_{t}\right\|^{2} + \beta[L^{2}_{i}L^{2}_{i}g^{2}\left(1+\frac{1}{N}\right)+12L^{2}_{i}L^{2}_{j}g^{2}\right] + \frac{6\beta S^{2}_{i}}{N}\sum_{n=1}^{N}\mathbb{E}\left\|x_{t}^{n}-\bar{x}_{t}\right\|^{2} \end{split}$$

where (a) holds due to $c_3 = \frac{\beta}{1-\beta}$. So $(1-\beta)(1+c_3) = 1$, and $(1+\frac{1}{c_3})\beta = 1$; (b) holds due to Lemma A.2 (d). When $\mod(t,q) \neq 0$, using $u_{s_tq}^n = \bar{u}_{s_tq}$, we have

$$\frac{1}{N}\sum_{n=1}^{N} \|u_t^n - \bar{u}_t\|^2 \le \sum_{s=s_tq}^{t-1} (1-\beta)^{t-1-s} \left[\beta [L_f^2 L_g^2 (1+\frac{1}{N}) + 12L_f^2 L_g^2] + \frac{6\beta S_F^2}{N} \sum_{n=1}^{N} \mathbb{E} \|x_s^n - \bar{x}_s\|^2 \right]$$
(30)

Summing (30) from $t = s_t q + 1$ to $(s_t + 1)q$, we have

$$\sum_{t=s_{t}q+1}^{(s_{t}+1)q} \frac{1}{N} \sum_{n=1}^{N} \|u_{t}^{n} - \bar{u}_{t}\|^{2} \\
\leq \sum_{t=s_{t}q+1}^{(s_{t}+1)q-1} \sum_{s=s_{t}q}^{t-1} (1-\beta)^{t-1-s} \left[\beta [L_{f}^{2} L_{g}^{2}(1+\frac{1}{N}) + 12L_{f}^{2} L_{g}^{2}] + \frac{6\beta S_{F}^{2}}{N} \sum_{n=1}^{N} \mathbb{E} \|x_{s}^{n} - \bar{x}_{s}\|^{2} \right] \\
\leq \sum_{t=s_{t}q+1}^{(s_{t}+1)q-1} (\sum_{s=0}^{q} (1-\beta)^{s}) \left[\beta [L_{f}^{2} L_{g}^{2}(1+\frac{1}{N}) + 12L_{f}^{2} L_{g}^{2}] + \frac{6\beta S_{F}^{2}}{N} \sum_{n=1}^{N} \mathbb{E} \|x_{t}^{n} - \bar{x}_{t}\|^{2} \right] \\
\leq \sum_{t=s_{t}q+1}^{(s_{t}+1)q-1} \left[[L_{f}^{2} L_{g}^{2}(1+\frac{1}{N}) + 12L_{f}^{2} L_{g}^{2}] + \frac{6S_{F}^{2}}{N} \sum_{n=1}^{N} \mathbb{E} \|x_{t}^{n} - \bar{x}_{t}\|^{2} \right]$$
(31)

where the last inequality holds due to $\sum_{s=0}^{q} (1-\beta)^s \leq \frac{1}{\beta}$. Similar to the (17), we have

$$\frac{1}{N}\sum_{n=1}^{N} \mathbb{E} \|x_t^n - \bar{x}_t\|^2 \le \frac{(q-1)\alpha^2}{N}\sum_{s=s_tq+1}^{t}\sum_{n=1}^{N} \mathbb{E} \|u_s^n - \bar{u}_s\|^2$$

By summing t from $s_t q + 1$ to $(s_t + 1)q$, we have

$$\begin{split} &\sum_{t=s_tq+1}^{(s_t+1)q} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \, \|x_t^n - \bar{x}_t\|^2 \\ \leq &\frac{(q-1)\alpha^2}{N} \sum_{t=s_tq+1}^{(s_t+1)q} \sum_{s=s_tq+1}^{t} \sum_{n=1}^{N} \mathbb{E} \|u_s^n - \bar{u}_s\|^2 \\ \leq &\frac{q(q-1)\alpha^2}{N} \sum_{t=s_tq+1}^{(s_t+1)q} \sum_{n=1}^{N} \mathbb{E} \|u_t^n - \bar{u}_t\|^2 \\ \leq &q(q-1)\alpha^2 \sum_{t=s_tq+1}^{(s_t+1)q-1} \left[[L_f^2 L_g^2 (1 + \frac{1}{N}) + 12L_f^2 L_g^2] + \frac{6S_F^2}{N} \sum_{n=1}^{N} \mathbb{E} \, \|x_t^n - \bar{x}_t\|^2 \right] \end{split}$$

where the last inequality holds due to (31). Rearrange the terms in the above inequality, we have

$$\frac{1 - 6q(q-1)\alpha^2 S_F^2}{N} \sum_{t=s_t q+1}^{(s_t+1)q} \sum_{n=1}^N \mathbb{E} \left\| x_t^n - \bar{x}_t \right\|^2 \le \sum_{t=s_t q+1}^{(s_t+1)q-1} q(q-1)\alpha^2 [L_f^2 L_g^2 (1+\frac{1}{N}) + 12L_f^2 L_g^2]$$

Assume $\alpha \leq \frac{1}{6qS_F}$, we have $\frac{1-6q(q-1)\alpha^2 S_F^2}{N} \geq \frac{5}{6N}$, then by summing the t from 0 to T - 1, we get the final result.

C.2 Proof of Theorem 4.4

Based on aforementioned lemmas, we are ready to prove the convergence of Theorem

Proof. Based on the C.1, we have

$$\begin{split} &\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla F(\bar{x}_t)\|^2 \\ \leq &2\frac{F(\bar{x}_0)-F(\bar{x}_T)}{\alpha T}-(1-\alpha S_F)\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\bar{u}_{t+1}\|^2+\frac{8S_F^2}{N}\frac{1}{T}\sum_{t=0}^{T-1}\sum_{n=1}^N\mathbb{E}\|x_t^n-\bar{x}_t\|^2 \\ &+\frac{4L_g^2S_f^2\sigma_g^2}{m}+\frac{4}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla \hat{F}(\bar{x}_t)-\bar{u}_{t+1}\|^2 \\ \leq &2\frac{F(\bar{x}_0)-F(\bar{x}_T)}{\alpha T}-(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta^2})\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\bar{u}_{t+1}\|^2+\frac{16S_F^2}{N}\frac{1}{T}\sum_{t=0}^{T-1}\sum_{n=1}^N\mathbb{E}\|x_t^n-\bar{x}_t\|^2 \\ &+\frac{4L_g^2S_f^2\sigma_g^2}{m}+\frac{8}{\beta T}\mathbb{E}\left\|\bar{u}_1-\nabla \hat{F}(\bar{x}_0)\right\|^2+\frac{8\beta L_f^2L_g^2}{N} \\ \leq &2\frac{F(\bar{x}_0)-F(\bar{x}_T)}{\alpha T}-(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta^2})\frac{1}{T}\sum_{t=0}^{T-1}\|\bar{u}_{t+1}\|^2 \\ &+\frac{96S_F^2}{5}q(q-1)\alpha^2[L_f^2L_g^2(1+\frac{1}{N})+12L_f^2L_g^2]+\frac{4L_g^2S_f^2\sigma_g^2}{m}+\frac{8}{\beta T}\mathbb{E}\left\|\bar{u}_1-\nabla \hat{F}(\bar{x}_0)\right\|^2+\frac{8\beta L_f^2L_g^2}{N} \\ \leq &2\frac{F(\bar{x}_0)-F(\bar{x}_T)}{\alpha T}-(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta^2})\frac{1}{T}\sum_{t=0}^{T-1}\|\bar{u}_{t+1}\|^2 \\ &+\frac{96S_F^2}{5}q(q-1)\alpha^2[L_f^2L_g^2(1+\frac{1}{N})+12L_f^2L_g^2]+\frac{4L_g^2S_f^2\sigma_g^2}{m}+\frac{8L_f^2L_g^2}{\beta BT}+\frac{8\beta L_f^2L_g^2}{N} \\ \leq &2\frac{F(\bar{x}_0)-F(\bar{x}_T)}{\alpha T}-(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta^2})\frac{1}{T}\sum_{t=0}^{T-1}\|\bar{u}_{t+1}\|^2 \\ &+\frac{96S_F^2}{5}q(q-1)\alpha^2[L_f^2L_g^2(1+\frac{1}{N})+12L_f^2L_g^2]+\frac{4L_g^2S_f^2\sigma_g^2}{m}+\frac{8L_f^2L_g^2}{\beta BT}+\frac{8\beta L_f^2L_g^2}{N} \\ \leq &2\frac{m}{2}\sum_{t=0}^{T}(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta^2})\frac{1}{T}\sum_{t=0}^{T-1}\|\bar{u}_{t+1}\|^2 \\ &+\frac{66S_F^2}{5}q(q-1)\alpha^2[L_f^2L_g^2(1+\frac{1}{N})+12L_f^2L_g^2]+\frac{4L_g^2S_f^2\sigma_g^2}{m}+\frac{8L_f^2L_g^2}{\beta BT}+\frac{8\beta L_f^2L_g^2}{N} \\ \leq &2\frac{F(\bar{x}_0)-F(\bar{x}_T)}{\alpha T}-(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta \beta^2})\frac{1}{T}\sum_{t=0}^{T-1}\|\bar{u}_{t+1}\|^2 \\ &+\frac{66S_F^2}{5}q(q-1)\alpha^2[L_f^2L_g^2(1+\frac{1}{N})+12L_f^2L_g^2]+\frac{6}{T}\sum_{t=0}^{T-1}\|\bar{u}_{t+1}\|^2 \\ &+\frac{6}{3}\sum_{t=0}^{T}(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta \beta^2})\frac{1}{T}\sum_{t=0}^{T-1}\|\bar{u}_{t+1}\|^2 \\ &+\frac{6}{3}\sum_{t=0}^{T}(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta \beta^2})\frac{1}{T}\sum_{t=0}^{T-1}\|\bar{u}_{t+1}\|^2 \\ &+\frac{6}{3}\sum_{t=0}^{T}(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta \beta^2})\frac{1}{T}\sum_{t=0}^{T-1}\|\bar{u}_{t+1}\|^2 \\ &+\frac{6}{3}\sum_{t=0}^{T}(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta \beta \gamma^2})\frac{1}{T}\sum_{t=0}^{T-1}\|\bar{u}_{t+1}\|^2 \\ &+\frac{6}{3}\sum_{t=0}^{T}\sum_{t=0}^{T}(1-\alpha S_F-\frac{16\alpha^2S_F^2}{\beta \gamma^2})\frac{1}{T}\sum$$

$$\begin{split} &\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla F(\bar{x}_t)\|^2 \\ \leq & 2\frac{F(\bar{x}_0) - F(\bar{x}_T)}{\alpha T} + \frac{96S_F^2}{5}q^2\alpha^2[L_f^2L_g^2(1+\frac{1}{N}) + 12L_f^2L_g^2] + \frac{4L_g^2S_f^2\sigma_g^2}{m} + \frac{8L_f^2L_g^2}{\beta BT} + \frac{8\beta L_f^2L_g^2}{N} \end{split}$$

We choose $h = O(1)$, $B = O(1)$, $\alpha = -\frac{1}{2}$, \sqrt{N} , $\alpha = (T/N^3)^{1/4}$ and $m = \varepsilon^{-2}$, we have

We choose b = O(1), B = O(1), $\alpha = \frac{1}{6S_F} \sqrt{\frac{N}{T}}$, $q = (T/N^3)^{1/4}$, and $m = \varepsilon^{-2}$, we have

$$\frac{1}{T}\sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \le \frac{12S_F[F(\bar{x}_0) - F(\bar{x}_*)]}{(NT)^{1/2}} + \frac{8[14L_f^2 L_g^2]}{3(NT)^{1/2}} + \frac{4L_g^2 S_f^2 \sigma_g^2}{m} + \frac{48L_f^2 L_g^2}{5(NT)^{1/2}} + \frac{20L_f^2 L_g^2}{3(NT)^{1/2}} + \frac{10L_f^2 L_g^2}{3(NT)^{1/2}} + \frac{10L_f^2$$

To let the right hand is less than ε^2 , we get $T = O(N^{-1}\varepsilon^{-4})$ and $\frac{T}{q} = (NT)^{3/4} = \varepsilon^{-3}$.

D Proof of Acc-FCSG-M Algorithm

D.1 Proofs of the Intermediate Lemmas

Lemma D.1. Suppose the sequence $\{x_t\}_0^T$ be generated from Acc-FCSG-M. We have

$$F(\bar{x}_{t+1}) \leq F(\bar{x}_t) - \left(\frac{\alpha}{2} - \frac{\alpha^2 S_F}{2}\right) \|\bar{u}_{t+1}\|^2 - \frac{\alpha}{2} \|\nabla F(\bar{x}_t)\|^2 + \frac{3\alpha S_F^2}{2N} \|x_t^n - \bar{x}_t\|^2 + \frac{3\alpha L_g^2 S_f^2 \sigma_g^2}{2m} + \frac{3\alpha}{2} \|\frac{1}{N} \sum_{n=1}^N \nabla \hat{F}^n(x_t) - \bar{u}_{t+1}\|^2$$
(32)

Proof.

$$F(\bar{x}_{t+1}) \stackrel{(a)}{\leq} F(\bar{x}_{t}) + \langle \nabla F(\bar{x}_{t}), \bar{x}_{t+1} - \bar{x}_{t} \rangle + \frac{S_{F}}{2} \|\bar{x}_{t+1} - \bar{x}_{t}\|^{2}$$

$$\stackrel{(b)}{=} F(\bar{x}_{t}) - \alpha \langle \nabla F(\bar{x}_{t}), \bar{u}_{t+1} \rangle + \frac{\alpha^{2} S_{F}}{2} \|\bar{u}_{t+1}\|^{2}$$

$$\stackrel{(c)}{=} F(\bar{x}_{t}) - (\frac{\alpha}{2} - \frac{\alpha^{2} S_{F}}{2}) \|\bar{u}_{t+1}\|^{2} - \frac{\alpha}{2} \|\nabla F(\bar{x}_{t})\|^{2} + \frac{\alpha}{2} \|\nabla F(\bar{x}_{t}) - \bar{u}_{t+1}\|^{2}$$

$$\leq F(\bar{x}_{t}) - (\frac{\alpha}{2} - \frac{\alpha^{2} S_{F}}{2}) \|\bar{u}_{t+1}\|^{2} - \frac{\alpha}{2} \|\nabla F(\bar{x}_{t})\|^{2} + \frac{3\alpha}{2} \|\nabla F(\bar{x}_{t}) - \frac{1}{N} \sum_{n=1}^{N} \nabla F^{n}(x_{t})\|^{2}$$

$$+ \frac{3\alpha}{2} \|\frac{1}{N} \sum_{n=1}^{N} \nabla F^{n}(x_{t}) - \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t})\|^{2} + \frac{3\alpha}{2} \|\frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}) - \bar{u}_{t+1}\|^{2}$$
(33)

where inequality (a) holds by the smoothness of F(x); equality (b) follows from update step in Step 9 of Acc-FCSG-M in Algorithm 2; (c) uses the fact that $\langle a, b \rangle = \frac{1}{2} [||a||^2 + ||a||^2 - ||a - b||^2]$. Taking expectation on both sides and considering the last third term

$$\mathbb{E} \|\nabla F(\bar{x}_t) - \frac{1}{N} \sum_{n=1}^N \nabla F^n(x_t) \|^2 \le \frac{1}{N} \sum_{n=1}^N \mathbb{E} \|\nabla F^n(\bar{x}_t) - \nabla F^n(x_t) \|^2 \\ \le \frac{S_F^2}{N} \sum_{n=1}^N \|x_t^n - \bar{x}_t\|^2$$
(34)

Considering the last second term, we have

$$\mathbb{E} \| \frac{1}{N} \sum_{n=1}^{N} \nabla F^{n}(x_{t}) - \frac{1}{N} \sum_{n=1}^{N} \nabla \hat{F}^{n}(x_{t}) \|^{2} \leq \frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \| \nabla F^{n}(x_{t}) - \nabla \hat{F}^{n}(x_{t}) \|^{2} \leq \frac{L_{g}^{2} S_{f}^{2} \sigma_{g}^{2}}{m}$$
(35)

Therefore, we obtain

$$F(\bar{x}_{t+1}) \leq F(\bar{x}_t) - \left(\frac{\alpha}{2} - \frac{\alpha^2 S_F}{2}\right) \|\bar{u}_{t+1}\|^2 - \frac{\alpha}{2} \|\nabla F(\bar{x}_t)\|^2 + \frac{3\alpha S_F^2}{2N} \|x_t^n - \bar{x}_t\|^2 + \frac{3\alpha L_g^2 S_f^2 \sigma_g^2}{2m} + \frac{3\alpha}{2} \|\frac{1}{N} \sum_{n=1}^N \nabla \hat{F}^n(x_t) - \bar{u}_{t+1}\|^2$$

$$(36)$$

Lemma D.2. (Lemma C.6 in [27]) Using the fact that $\mathbb{E}\bar{u}_{t+1} = \nabla \hat{F}(x_t)$ and \bar{u}_{t+1} is the momentumbased variance reduction estimator, we have

$$\mathbb{E}\|\bar{u}_{t+1} - \nabla\hat{F}(x_t)\|^2 \leq (1-\beta)^2 \mathbb{E}\|\bar{u}_t - \nabla\hat{F}(x_{t-1})\|^2 + \frac{8(1-\beta)^2 S_F^2}{N^2 b} \frac{q-1}{q} \sum_{n=1}^N \alpha^2 \mathbb{E}\|u_t^n - \bar{u}_t\|^2 + \frac{4(1-\beta)^2 S_F^2 \alpha^2}{N b} \mathbb{E}\|\bar{u}_t\|^2 + \frac{2\beta^2 L_f^2}{N b}$$
(37)

Lemma D.3. Assume that the sequences u_t are generated from Acc-FCSG-M, set $\gamma = \frac{1}{q}$ and $\alpha \leq \frac{1}{12S_Fq}$, and given that $\beta = c\alpha^2$, $c = \frac{30S_F^2}{bN}$ we have

$$\frac{15}{72N}\sum_{t=s_t}^{\bar{s}} \alpha \sum_{n=1}^{N} \mathbb{E}\|u_t^n - \bar{u}_t\|^2 \le \frac{1}{8}\sum_{t=s_t}^{\bar{s}} \alpha \mathbb{E}\|\bar{u}_t\|^2 + \left[\frac{L_f^2 c^2}{8bS_F^2} + \frac{3L_f^2 L_g^2 c^2}{8S_F^2}\right]\sum_{t=s_t}^{\bar{s}} \alpha^3 \qquad (38)$$

$$\begin{aligned} &Proof.\\ &\sum_{n=1}^{N} \mathbb{E} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} \leq \sum_{n=1}^{N} \mathbb{E} \left\| \frac{1}{b} \sum_{n=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) + (1-\beta)(u_{t}^{n} - \frac{1}{b} \sum_{n=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n})) \right. \\ &- \frac{1}{N} \sum_{k=1}^{N} \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k}) + (1-\beta)(u_{t}^{k} - \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t-1}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k})) \right] \right\|^{2} \\ &= \sum_{n=1}^{N} \mathbb{E} \left\| (1-\beta)(u_{t}^{n} - \bar{u}_{t}) + \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k}) \right. \\ &- (1-\beta) \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t-1}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k}) \right] \right\|^{2} \\ &\leq (1+\gamma)(1-\beta)^{2} \sum_{n=1}^{N} \mathbb{E} \|u_{t}^{n} - \bar{u}_{t}\|^{2} \\ &+ (1+\frac{1}{\gamma})\mathbb{E} \left\| \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k}) \right] \\ &- (1-\beta) \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k}) \right] \\ &- (1-\beta) \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t-1}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k}) \right] \right\|^{2} \end{aligned} \tag{39}$$

where the second inequality is due to Young's inequality. For the second term, we have

$$\begin{split} &\sum_{n=1}^{N} \mathbb{E} \left\| \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{i}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{k}^{k};\xi_{k,i}^{k},\mathcal{B}_{t,i}^{k}) \right] \\ &- (1-\beta) \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) \right] \right\|^{2} \\ &= \sum_{n=1}^{N} \mathbb{E} \left\| \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{i}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t}^{k};\xi_{k}^{k},\mathcal{B}_{t,i}^{k}) \right] - \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t}^{k};\xi_{k}^{k},\mathcal{B}_{t,i}^{k}) \right] \\ &- \frac{1}{N} \sum_{k=1}^{n} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t-1}^{k};\xi_{k}^{k},\mathcal{B}_{t,i}^{k}) \right] + \beta \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t-1}^{k};\xi_{k}^{k},\mathcal{B}_{t,i}^{k}) \right] \\ &- \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t-1}^{k};\xi_{k}^{k},\mathcal{B}_{t,i}^{k}) \right] \right\|^{2} \\ &\leq 2 \sum_{n=1}^{N} \mathbb{E} \left\| \left\| \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t-1}^{k};\xi_{k}^{k},\mathcal{B}_{t,i}^{k}) \right] \right\|^{2} \\ &+ 2\beta^{2} \sum_{n=1}^{N} \mathbb{E} \left\| \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) \right\|^{2} \\ &+ 2\beta^{2} \sum_{n=1}^{N} \mathbb{E} \left\| \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{n}) \right\|^{2} \\ &+ 2\beta^{2} \sum_{n=1}^{N} \mathbb{E} \left\| \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{b} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t-1}^{k};\xi_{k,i}^{k},\mathcal{B}_{k,i}^{k}) \right\|^{2} \\ & \begin{pmatrix} (^{b} 2S_{F}^{2} \sum_{n=1}^{N} \mathbb{E} \left\| \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{n$$

where (a) holds due to Lemma A.1, and (b) uses Lemma A.2 (b). For the last term in (40), we have

$$\sum_{n=1}^{N} \mathbb{E} \left\| \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t-1}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k}) \right\|^{2}$$

$$= \sum_{n=1}^{N} \mathbb{E} \left\| \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \nabla \hat{F}^{n}(x_{t-1}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \left[\frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{k}(x_{t-1}^{k};\xi_{t,i}^{k},\mathcal{B}_{t,i}^{k}) - \nabla F(x_{t-1}^{k}) \right] \right\|^{2}$$

$$+ \left[\nabla \hat{F}^{n}(x_{t-1}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \nabla \hat{F}^{k}(x_{t-1}^{k}) \right] \right\|^{2}$$

$$\leq 2 \sum_{n=1}^{N} \mathbb{E} \left\| \frac{1}{b} \sum_{i=1}^{b} \nabla \hat{F}^{n}(x_{t-1}^{n};\xi_{t,i}^{n},\mathcal{B}_{t,i}^{n}) - \nabla \hat{F}^{n}(x_{t-1}^{n}) \right\|^{2} + 2 \sum_{n=1}^{N} \mathbb{E} \| \hat{F}^{n}(x_{t-1}^{n}) - \frac{1}{N} \sum_{k=1}^{N} \nabla \hat{F}^{k}(x_{t-1}^{k}) \|^{2}$$

$$\leq 26 N L_{f}^{2} L_{g}^{2} + 12 S_{F}^{2} \sum_{n=1}^{N} \mathbb{E} \| x_{t-1}^{n} - \bar{x}_{t-1} \|^{2}$$

$$(41)$$

where the first inequality is due to Lemma A.1 and the last inequality is due to Lemma A.2 (c) and Lemma A.3. Therefore, by combining above inequalities (39), (40) and (41), when $\mod(t+1,q) \neq 0$ we have

$$\begin{split} \sum_{n=1}^{N} \mathbb{E} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} \\ &\leq (1-\beta)^{2}(1+\gamma) \sum_{n=1}^{N} \mathbb{E} \|u_{t}^{n} - \bar{u}_{t}\|^{2} + 2S_{F}^{2}(1+\frac{1}{\gamma}) \sum_{n=1}^{N} \mathbb{E} \|x_{t}^{n} - x_{t-1}^{n}\|^{2} \\ &+ 52NL_{f}^{2}L_{g}^{2}(1+\frac{1}{\gamma})\beta^{2} + 24S_{F}^{2}(1+\frac{1}{\gamma})\beta^{2} \sum_{n=1}^{N} \mathbb{E} \|x_{t-1}^{n} - \bar{x}_{t-1}\|^{2} \\ \overset{(a)}{\leq} (1-\beta)^{2}(1+\gamma) \sum_{n=1}^{N} \mathbb{E} \|u_{t}^{n} - \bar{u}_{t}\|^{2} + 2S_{F}^{2}(1+\frac{1}{\gamma}) \sum_{n=1}^{N} \mathbb{E} \|\alpha u_{t}^{n}\|^{2} \\ &+ 52NL_{f}^{2}L_{g}^{2}(1+\frac{1}{\gamma})\beta^{2} + 24S_{F}^{2}(1+\frac{1}{\gamma})\beta^{2}(q-1) \sum_{s=s_{t}q+1}^{t-1} \alpha^{2} \sum_{n=1}^{N} \mathbb{E} \|u_{s}^{n} - \bar{u}_{s}\|^{2} \\ &\leq (1-\beta)^{2}(1+\gamma) \sum_{n=1}^{N} \mathbb{E} \|u_{t}^{n} - \bar{u}_{t}\|^{2} + 4S_{F}^{2}(1+\frac{1}{\gamma}) \sum_{n=1}^{N} \mathbb{E} [\|\alpha(u_{t}^{n} - \bar{u}_{t})\|^{2} + \|\alpha \bar{u}_{t}\|^{2}] \\ &+ 52NL_{f}^{2}L_{g}^{2}(1+\frac{1}{\gamma})\beta^{2} + 24S_{F}^{2}(1+\frac{1}{\gamma})\beta^{2}(q-1) \sum_{s=s_{t}q+1}^{t-1} \alpha^{2} \sum_{n=1}^{N} \mathbb{E} \|u_{s}^{n} - \bar{u}_{s}\|^{2} \end{split}$$
(42)

where (a) is due to (17). Then we have

$$\sum_{n=1}^{N} \mathbb{E} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} = \left[(1-\beta)^{2}(1+\gamma) + 4S_{F}^{2}(1+\frac{1}{\gamma})\alpha^{2}\right] \sum_{n=1}^{N} \mathbb{E} \|u_{t}^{n} - \bar{u}_{t}\|^{2} + 4NS_{F}^{2}(1+\frac{1}{\gamma})\alpha^{2} \mathbb{E} \|\bar{u}_{t}\|^{2} + 52NL_{f}^{2}L_{g}^{2}(1+\frac{1}{\gamma})\beta^{2} + 24S_{F}^{2}(1+\frac{1}{\gamma})\beta^{2}(q-1) \sum_{s=s_{t}q+1}^{t-1} \alpha^{2} \sum_{n=1}^{N} \mathbb{E} \|u_{s}^{n} - \bar{u}_{s}\|^{2}$$
(43)

Set $\gamma = \frac{1}{q}$ and $\alpha \le \frac{1}{12S_F q}$, and given that $\beta \in (0, 1)$, $(1 - \beta)^2 (1 + \gamma) + 4S_F^2 (1 + \frac{1}{\gamma}) \alpha^2 \le 1 + \frac{1}{q} + 4S_F^2 (1 + q) \alpha^2 \le 1 + \frac{1}{q} + \frac{q + 1}{36q^2} \le 1 + \frac{19}{18q}$ (44) Putting the (44) in (43), and considering $\gamma = \frac{1}{q}$ and $\alpha \leq \frac{1}{12S_Fq}$, $\beta = c\alpha^2$, we have

$$\sum_{n=1}^{N} \mathbb{E} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2}$$

$$\leq (1 + \frac{19}{18q}) \sum_{n=1}^{N} \mathbb{E} \|u_{t}^{n} - \bar{u}_{t}\|^{2} + 4NS_{F}^{2}(1 + \frac{1}{\gamma})\alpha^{2} \mathbb{E} \|\bar{u}_{t}\|^{2}$$

$$+ 52NL_{f}^{2}L_{g}^{2}(1 + \frac{1}{\gamma})\beta^{2} + 24S_{F}^{2}(1 + \frac{1}{\gamma})\beta^{2}(q - 1) \sum_{s=s_{t}q+1}^{t-1} \alpha^{2} \sum_{n=1}^{N} \mathbb{E} \|u_{s}^{n} - \bar{u}_{s}\|^{2}$$

$$\leq (1 + \frac{19}{18q}) \sum_{n=1}^{N} \mathbb{E} \|u_{t}^{n} - \bar{u}_{t}\|^{2} + \frac{2NS_{F}}{3} \alpha \mathbb{E} \|\bar{u}_{t}\|^{2}$$

$$+ \frac{26NL_{f}^{2}L_{g}^{2}c^{2}}{3S_{F}} \alpha^{3} + 24S_{F}^{2}q^{2}c^{2}\alpha^{4} \sum_{s=s_{t}q+1}^{t-1} \alpha^{2} \sum_{n=1}^{N} \mathbb{E} \|u_{s}^{n} - \bar{u}_{s}\|^{2}$$
(45)

We know that when $\mod(t,q) = 0$ (i.e. $t = s_t q$), $\sum_{n=1}^{N} ||u_t^n - \bar{u}_t||^2 = 0$

$$\begin{split} &\sum_{n=1}^{N} \mathbb{E} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} \\ &\leq \frac{2NS_{F}}{3} \sum_{s=s_{t}q}^{t} (1 + \frac{19}{18q})^{t-s} \alpha \mathbb{E} \|\bar{u}_{s}\|^{2} + \frac{26NL_{f}^{2}L_{g}^{2}c^{2}}{3S_{F}} \sum_{s=s_{t}q}^{t} (1 + \frac{19}{18q})^{t-s} \alpha^{3} \\ &+ 24S_{F}^{2}q^{2}c^{2} \sum_{s=s_{t}}^{t} (1 + \frac{19}{18q})^{t-s} \alpha^{4} \sum_{\bar{s}=s_{t}q}^{s} \alpha^{2} \sum_{n=1}^{N} \mathbb{E} \|u_{\bar{s}}^{n} - \bar{u}_{\bar{s}}\|^{2} \\ &\leq \frac{2NS_{F}}{3} \sum_{s=s_{t}q}^{t} (1 + \frac{19}{18q})^{q} \alpha \mathbb{E} \|\bar{u}_{s}\|^{2} + \frac{26NL_{f}^{2}L_{g}^{2}c^{2}}{3S_{F}} \sum_{s=s_{t}q}^{t} \left(1 + \frac{19}{18q}\right)^{q} \alpha^{3} \\ &+ 24S_{F}^{2}q^{3}c^{2} (\frac{1}{12S_{F}q})^{5} (1 + \frac{19}{18q})^{q} \sum_{s=s_{t}q}^{t} \alpha \sum_{n=1}^{N} \mathbb{E} \|u_{s}^{n} - \bar{u}_{s}\|^{2} \\ &\leq 2NS_{F} \sum_{s=s_{t}q}^{t} \alpha \mathbb{E} \|\bar{u}_{s}\|^{2} + \frac{26NL_{f}^{2}L_{g}^{2}c^{2}}{S_{F}} \sum_{s=s_{t}q}^{t} \alpha^{3} + 72S_{F}^{2}q^{3}c^{2} (\frac{1}{12S_{F}q})^{5} \sum_{s=s_{t}q}^{t} \alpha \sum_{n=1}^{N} \mathbb{E} \|u_{s}^{n} - \bar{u}_{s}\|^{2} \end{split}$$

where the third inequality is due to $(1 + 19/18q)^q \le e^{19/18} \le 3$. Multiplying α on both side and summing over $[s_tq,\bar{s})$ in one inner loop, where $\bar{s} = (s_t + 1)q$ we have

$$\begin{split} &\sum_{t=s_{tq}}^{\bar{s}} \alpha \sum_{n=1}^{N} \mathbb{E} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} \leq 2NS_{F} \sum_{t=s_{tq}}^{\bar{s}} \alpha \sum_{s=s_{tq}}^{t} \alpha \mathbb{E} \|\bar{u}_{s}\|^{2} + \frac{26NL_{f}^{2}L_{g}^{2}c^{2}}{S_{F}} \sum_{t=s_{tq}}^{\bar{s}} \alpha \sum_{s=s_{tq}}^{t} \alpha^{3} \\ &+ 72S_{F}^{2}q^{3}c^{2}(\frac{1}{12S_{F}q})^{5} \sum_{t=s_{tq}}^{\bar{s}} \alpha \sum_{s=s_{tq}}^{t} \alpha \sum_{n=1}^{N} \mathbb{E} \|u_{s}^{n} - \bar{u}_{s}\|^{2} \\ &\leq 2NS_{F}(\sum_{t=s_{tq}}^{\bar{s}} \alpha) \sum_{t=s_{tq}}^{\bar{s}} \alpha \mathbb{E} \|\bar{u}_{t}\|^{2} + \frac{26NL_{f}^{2}L_{g}^{2}c^{2}}{S_{F}} (\sum_{t=s_{tq}}^{\bar{s}} \alpha) \sum_{t=s_{tq}}^{\bar{s}} \alpha^{3} \\ &+ 72S_{F}^{2}q^{3}c^{2}(\frac{1}{12S_{F}q})^{5}(\sum_{t=s_{t}}^{\bar{s}} \alpha) \sum_{t=s_{tq}}^{\bar{s}} \alpha \sum_{n=1}^{N} \mathbb{E} \|u_{t}^{n} - \bar{u}_{t}\|^{2} \\ &\leq \frac{N}{6} \sum_{t=s_{tq}}^{\bar{s}} \alpha \mathbb{E} \|\bar{u}_{t}\|^{2} + \frac{13NL_{f}^{2}L_{g}^{2}c^{2}}{6S_{F}^{2}} \sum_{t=s_{tq}}^{\bar{s}} \alpha^{3} + 72S_{F}^{2}q^{4}c^{2}(\frac{1}{12S_{F}q})^{6} \sum_{t=s_{tq}}^{\bar{s}+1} \alpha \sum_{n=1}^{N} \mathbb{E} \|u_{t}^{n} - \bar{u}_{t}\|^{2} \end{split}$$

Rearranging the terms, we get,

$$[1 - 72S_F^2 q^4 c^2 (\frac{1}{12S_F q})^6] \sum_{t=s_t q+1}^{\bar{s}+1} \alpha \sum_{n=1}^N \mathbb{E} \|u_t^n - \bar{u}_t\|^2 \leq \frac{N}{6} \sum_{t=s_t q}^{\bar{s}} \alpha \mathbb{E} \|\bar{u}_t\|^2 + \frac{13NL_f^2 L_g^2 c^2}{6S_F^2} \sum_{t=s_t q}^{\bar{s}} \alpha^3 (1 - 1) \sum_{t=s_t q+1}^{\bar{s}+1} \alpha \sum_{n=1}^N \mathbb{E} \|u_t^n - \bar{u}_t\|^2 \leq \frac{N}{6} \sum_{t=s_t q}^{\bar{s}} \alpha \mathbb{E} \|\bar{u}_t\|^2 + \frac{13NL_f^2 L_g^2 c^2}{6S_F^2} \sum_{t=s_t q}^{\bar{s}} \alpha^3 (1 - 1) \sum_{t=s_t q+1}^{\bar{s}+1} \alpha \sum_{n=1}^N \mathbb{E} \|u_t^n - \bar{u}_t\|^2 \leq \frac{N}{6} \sum_{t=s_t q}^{\bar{s}} \alpha \mathbb{E} \|\bar{u}_t\|^2 + \frac{13NL_f^2 L_g^2 c^2}{6S_F^2} \sum_{t=s_t q}^{\bar{s}} \alpha^3 (1 - 1) \sum_{t=s_t q+1}^{\bar{s}+1} \alpha \sum_{n=1}^N \mathbb{E} \|u_t^n - \bar{u}_t\|^2 \leq \frac{N}{6} \sum_{t=s_t q+1}^{\bar{s}} \alpha \mathbb{E} \|\bar{u}_t\|^2 + \frac{13NL_f^2 L_g^2 c^2}{6S_F^2} \sum_{t=s_t q+1}^{\bar{s}} \alpha^3 (1 - 1) \sum_{t=s_t q+1}^{\bar{s}+1} \alpha \sum_{t=s_t q+1}^{\bar{s}+1} \alpha$$

Given that $c = \frac{30S_F^2}{bN}$, and $(1 - 72S_F^2q^4c^2(\frac{1}{12S_Fq})^6)/2 \ge \frac{101}{240}$. By multiply $\frac{1}{2N}$ on both size and summing t from 1 to T, we have

$$\frac{101}{240N} \sum_{t=1}^{T} \alpha \sum_{n=1}^{N} \mathbb{E} \|u_t^n - \bar{u}_t\|^2 \le \frac{1}{12} \sum_{t=1}^{T} \alpha \mathbb{E} \|\bar{u}_t\|^2 + \frac{13L_f^2 L_g^2 c^2}{12S_F^2} T\alpha^3$$
(46)

D.2 Proof of Theorem 4.7

In this section, we show the Proof of Theorem 4.7.

Proof. Set $\alpha = \frac{1}{12qS_F}$, $\beta = c \cdot \alpha^2$, $c = \frac{30S_F^2}{bN}$. Recall Lemma D.2, we have

$$\mathbb{E}\|\bar{u}_{t+1} - \nabla\hat{F}(x_t)\|^2 \\
\leq (1-\beta)^2 \mathbb{E}\|\bar{u}_t - \nabla\hat{F}(x_{t-1})\|^2 + \frac{8(1-\beta)^2 S_F^2}{N^2 b} \frac{q-1}{q} \sum_{n=1}^N \alpha^2 \mathbb{E}\|u_t^n - \bar{u}_t\|^2 \\
+ \frac{4(1-\beta)^2 S_F^2 \alpha^2}{N b} \mathbb{E}\|\bar{u}_t\|^2 + \frac{2\beta^2 L_f^2}{N b} \\
\leq (1-\beta) \mathbb{E}\|\bar{u}_t - \nabla\hat{F}(x_{t-1})\|^2 + \frac{8S_F^2}{N^2 b} \sum_{n=1}^N \alpha^2 \mathbb{E}\|u_t^n - \bar{u}_t\|^2 + \frac{4S_F^2 \alpha^2}{N b} \mathbb{E}\|\bar{u}_t\|^2 + \frac{2\beta^2 L_f^2}{N b} \tag{47}$$

We define the potential function as a linear combination of the objective function and the gradient estimation error:

$$\Phi_t = F(\bar{x}_t) + \frac{3\alpha}{2\beta} \|\bar{u}_{t+1} - \nabla \hat{F}(x_t)\|^2$$
(48)

Therefore, we have

$$\begin{split} & \mathbb{E}\Phi_{t+1} - \mathbb{E}\Phi_t = \mathbb{E}[F(\bar{x}_{t+1}) + \frac{3\alpha}{2\beta} \|\bar{u}_{t+2} - \nabla\hat{F}(x_{t+1})\|^2] - \mathbb{E}[F(\bar{x}_t) + \frac{3\alpha}{2\beta} \|\bar{u}_{t+1} - \nabla\hat{F}(x_t)\|^2] \\ & \leq -\left(\frac{\alpha}{2} - \frac{\alpha^2 S_F}{2}\right) \mathbb{E}\|\bar{u}_{t+1}\|^2 - \frac{\alpha}{2} \mathbb{E}\|\nabla F(\bar{x}_t)\|^2 + \frac{3\alpha S_F^2}{2N} \mathbb{E}\|x_t^n - \bar{x}_t\|^2 + \frac{3\alpha L_g^2 S_f^2 \sigma_g^2}{2m} \\ & + \frac{3\alpha}{2} \mathbb{E}\|\frac{1}{N} \sum_{n=1}^N \nabla\hat{F}^n(x_t) - \bar{u}_{t+1}\|^2 - \frac{3\alpha}{2} \mathbb{E}\|\bar{u}_{t+1} - \frac{1}{N} \sum_{n=1}^N \nabla\hat{F}^n(x_t^n)\|^2 \\ & + \frac{12\alpha S_F^2}{N^2 b\beta} \sum_{n=1}^N \alpha^2 \mathbb{E}\|u_{t+1}^n - \bar{u}_{t+1}\|^2 + \frac{6S_F^2 \alpha^3}{Nb\beta} \mathbb{E}\|\bar{u}_{t+1}\|^2 + \frac{3\alpha\beta L_f^2}{Nb} \end{split}$$
(49)

Rearranging (49) and taking the telescoping sum over t in $[s_tq, \bar{s})$ in one inner loop, where $\bar{s} = (s_t + 1)q$, we have

$$\begin{split} &\sum_{t=s_{t}q}^{\bar{s}-1} \frac{\alpha}{2} \mathbb{E} \|\nabla F(\bar{x}_{t})\|^{2} \\ &\leq \sum_{t=s_{t}q}^{\bar{s}-1} [\mathbb{E}\Phi_{t} - \mathbb{E}\Phi_{t+1}] - (\frac{\alpha}{2} - \frac{\alpha^{2}S_{F}}{2} - \frac{6S_{F}^{2}\alpha^{3}}{Nb\beta}) \sum_{t=s_{t}q}^{\bar{s}-1} \mathbb{E} \|\bar{u}_{t+1}\|^{2} + \frac{3\alpha S_{F}^{2}}{2N} \sum_{t=s_{t}q}^{\bar{s}-1} \mathbb{E} \|x_{t}^{n} - \bar{x}_{t}\|^{2} \\ &+ \sum_{t=s_{t}q}^{\bar{s}-1} \frac{3\alpha L_{g}^{2}S_{f}^{2}\sigma_{g}^{2}}{2m} + \frac{12\alpha S_{F}^{2}}{N^{2}b\beta} \sum_{t=s_{t}q}^{\bar{s}-1} \sum_{n=1}^{N} \alpha^{2} \mathbb{E} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} + \sum_{t=s_{t}q}^{\bar{s}-1} \frac{3\alpha\beta L_{f}^{2}}{Nb} \\ &\leq \sum_{t=s_{t}q}^{\bar{s}-1} [\mathbb{E}\Phi_{t} - \mathbb{E}\Phi_{t+1}] - (\frac{\alpha}{2} - \frac{\alpha^{2}S_{F}}{2} - \frac{6S_{F}^{2}\alpha^{3}}{Nb\beta}) \sum_{t=s_{t}q}^{\bar{s}-1} \mathbb{E} \|\bar{u}_{t+1}\|^{2} \\ &+ \frac{3\alpha S_{F}^{2}}{2N} \sum_{t=s_{t}q}^{\bar{s}-1} \mathbb{E} [(q-1)\alpha^{2} \sum_{s=s_{t}q}^{t} \sum_{n=1}^{N} \|u_{s}^{n} - \bar{u}_{s}\|^{2}] + \sum_{t=s_{t}q}^{\bar{s}-1} \frac{3\alpha L_{g}^{2}S_{f}^{2}\sigma_{g}^{2}}{2m} \\ &+ \frac{12\alpha S_{F}^{2}}{N^{2}b\beta} \sum_{t=s_{t}q}^{\bar{s}-1} \sum_{n=1}^{N} \alpha^{2} \mathbb{E} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} + \sum_{t=s_{t}q}^{\bar{s}-1} \frac{3\alpha\beta L_{f}^{2}}{Nb} \end{split}$$

$$\tag{50}$$

where the last inequality uses (17). Furthermore,

$$\begin{split} &\sum_{t=s_{tq}}^{\tilde{s}-1} \frac{\alpha}{2} \|\nabla F(\bar{x}_{t})\|^{2} \leq \sum_{t=s_{tq}}^{\tilde{s}-1} [\mathbb{E}\Phi_{t} - \mathbb{E}\Phi_{t+1}] - (\frac{\alpha}{2} - \frac{\alpha^{2}S_{F}}{2} - \frac{6S_{F}^{2}\alpha^{3}}{Nb\beta}) \sum_{t=s_{tq}}^{\tilde{s}-1} \mathbb{E}\|\bar{u}_{t+1}\|^{2} \\ &+ \sum_{t=s_{tq}}^{\tilde{s}-1} \frac{3\alpha L_{g}^{2}S_{f}^{2}\sigma_{g}^{2}}{2m} + \frac{3S_{F}^{2}\alpha}{2N} [(q-1) \times q \times \frac{1}{12S_{F}q} \times \frac{1}{12S_{F}q} \sum_{t=s_{tq}}^{\tilde{s}-1} \sum_{n=1}^{N} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2}] \\ &+ \sum_{t=s_{tq}}^{\tilde{s}-1} [\frac{12S_{F}^{2}\alpha^{3}}{N^{2}b\beta} \sum_{n=1}^{N} \mathbb{E}\|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} + \frac{3\alpha\beta L_{f}^{2}}{Nb}] \\ &\leq \sum_{t=s_{tq}}^{\tilde{s}-1} [\frac{12S_{F}^{2}\alpha^{3}}{N^{2}b\beta} \sum_{n=1}^{N} \mathbb{E}\|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} + \frac{3\alpha\beta L_{f}^{2}}{2}] \\ &+ \frac{\alpha}{96N} \sum_{t=s_{tq}}^{\tilde{s}-1} \sum_{n=1}^{N} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} + \sum_{t=s_{tq}}^{\tilde{s}-1} [\frac{12S_{F}^{2}\alpha^{3}}{Nb\beta}) \sum_{n=1}^{N} \mathbb{E}\|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} + \frac{3\alpha\beta L_{g}^{2}}{2m} \\ &+ \frac{\alpha}{96N} \sum_{t=s_{tq}}^{\tilde{s}-1} \sum_{n=1}^{N} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} + \sum_{t=s_{tq}}^{\tilde{s}-1} [\frac{12S_{F}^{2}\alpha^{3}}{N^{2}b\beta} \sum_{n=1}^{N} \mathbb{E}\|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} + \frac{3\alpha\beta L_{g}^{2}S_{f}^{2}\sigma_{g}^{2}}{2m} \\ &+ \frac{\alpha}{96N} \sum_{t=s_{tq}}^{\tilde{s}-1} \sum_{n=1}^{N} \|u_{t+1}^{n} - \bar{u}_{t+1}\|^{2} + \sum_{t=s_{tq}}^{\tilde{s}-1} [\frac{12S_{F}^{2}\alpha^{3}}{N^{2}b\beta} \sum_{n=1}^{N} \mathbb{E}\|u_{t+1}\|^{2} + \frac{3\alpha\beta L_{g}^{2}S_{f}^{2}\sigma_{g}^{2}}{2m} \\ &+ (\frac{1}{96N} + \frac{12S_{F}^{2}}{N^{2}bc})\alpha[\sum_{t=s_{tq}}^{\tilde{s}-1} \sum_{n=1}^{N} \|u_{s+1}^{n} - \bar{u}_{s+1}\|^{2}] + \sum_{t=s_{tq}}^{\tilde{s}-1} \frac{3\alpha L_{g}^{2}S_{f}^{2}\sigma_{g}^{2}}{2m} \\ &+ (\frac{1}{96N} + \frac{12S_{F}^{2}}{N^{2}bc})\alpha[\sum_{t=s_{tq}}^{\tilde{s}-1} \sum_{n=1}^{N} \|u_{s+1}^{n} - \bar{u}_{s+1}\|^{2}] + \sum_{t=s_{tq}}^{\tilde{s}-1} \frac{3\beta\alpha L_{f}^{2}}{Nb} \\ &= \sum_{t=s_{tq}}^{\tilde{s}-1} \mathbb{E}[\Phi_{t} - \Phi_{t+1}] - (\frac{3\alpha}{10} - \frac{\alpha^{2}S_{F}}{2}) \sum_{t=s_{tq}}^{\tilde{s}-1} \mathbb{E}\|\bar{u}_{t+1}\|^{2} + \frac{101\alpha}{240N} \sum_{t=s_{tq}}^{\tilde{s}-1} \sum_{n=1}^{N} \|u_{s}^{n} - \bar{u}_{s}\|^{2} \\ &+ \sum_{t=s_{tq}}^{\tilde{s}-1} \frac{3\alpha L_{g}^{2}S_{f}^{2}\sigma_{g}^{2}}{2m} + \sum_{t=s_{tq}}^{\tilde{s}-1} \frac{3\beta\alpha L_{f}^{2}}{Nb} \\ \end{split}$$

where the last equality holds by to $c = \frac{30S_F^2}{bN}$. Therefore, by summing t from 0 to T, we have

$$\begin{split} &\sum_{t=0}^{T-1} \frac{\alpha}{2} \|\nabla F(\bar{x}_t)\|^2 \\ \leq &\mathbb{E}[\Phi_0 - \Phi_T] - (\frac{3\alpha}{10} - \frac{\alpha^2 S_F}{2}) \sum_{t=0}^{T-1} \mathbb{E}\|\bar{u}_{t+1}\|^2 + \frac{101\alpha}{240N} \sum_{t=0}^{T-1} \sum_{n=1}^N \|u_s^n - \bar{u}_s\|^2 \\ &+ \frac{3\alpha L_g^2 S_f^2 \sigma_g^2 T}{2m} + \frac{3\beta\alpha L_f^2 T}{Nb} \\ \leq &\mathbb{E}[\Phi_0 - \Phi_T] - (\frac{3\alpha}{10} - \frac{\alpha}{12} - \frac{\alpha^2 S_F}{2}) \sum_{t=0}^{T-1} \mathbb{E}\|\bar{u}_{t+1}\|^2 + \frac{13L_f^2 L_g^2 c^2}{12S_F^2} T\alpha^3 \\ &+ \frac{3\alpha L_g^2 S_f^2 \sigma_g^2 T}{2m} + \frac{3\beta\alpha L_f^2 T}{Nb} \\ \leq &\mathbb{E}[\Phi_0 - \Phi_T] + \frac{13L_f^2 L_g^2 c^2}{12S_F^2} T\alpha^3 + \frac{3\alpha L_g^2 S_f^2 \sigma_g^2 T}{2m} + \frac{3\beta\alpha L_f^2 T}{Nb} \\ \leq &\mathbb{E}[F(\bar{x}_0) - F(\bar{x}_T)] + [\frac{3\alpha}{2\beta} \mathbb{E}\|\bar{u}_1 - \frac{1}{N} \sum_{n=1}^N \nabla \hat{F}^n(x_0^n)\|^2 + \frac{13L_f^2 L_g^2 c^2}{12S_F^2} T\alpha^3 \\ &+ \frac{3\alpha L_g^2 S_f^2 \sigma_g^2 T}{2m} + \frac{3\beta\alpha L_f^2 T}{Nb} \end{split}$$

where (a) holds due to D.3. Therefore,

$$\frac{1}{T}\sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \le \frac{2[F(\bar{x}_0) - F(\bar{x}_T)]}{T\alpha} + \frac{3L_f^2 L_g^2}{\beta BNT} + \frac{13L_f^2 L_g^2 c^2}{6S_F^2}\alpha^2 + \frac{3L_g^2 S_f^2 \sigma_g^2}{m} + \frac{6\beta L_f^2}{Nb}$$

let b as $O(1)(b \ge 1)$, and choose $q = (T/N^2)^{1/3}$. Therefore, $\alpha = \frac{1}{12qS_F} = \frac{N^{2/3}}{12S_F T^{1/3}}$, since $c = \frac{30S_F^2}{bN}$, we have $\beta = c\alpha^2 = \frac{5N^{1/3}}{24T^{2/3}b}$. And let $B = \frac{T^{1/3}}{N^{2/3}}$

Therefore, we have

$$\frac{1}{T} \sum_{t=0}^{T-1} \|\nabla F(\bar{x}_t)\|^2 \\
\leq \frac{24S_F[F(\bar{x}_0) - F(\bar{x}_*)]}{(NT)^{2/3}} + \frac{72L_f^2 L_g^2 b}{5(NT)^{2/3}} + \frac{325L_f^2 L_g^2}{24b^2(TN)^{2/3}} + \frac{3L_g^2 S_f^2 \sigma_g^2}{m} + \frac{5L_f^2}{4b^2(NT)^{2/3}} \tag{51}$$

To let the right hand less than ε^2 when $m = \varepsilon^{-2}$ and b = O(1), we get $T = O(N^{-1}\varepsilon^{-3})$ and $\frac{T}{q} = (NT)^{2/3} = \varepsilon^{-2}$.